STUDENT NUMBER: ______________________________

QUEEN'S UNIVERSITY
FACULTY OF ENGINEERING AND APPLIED SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
APSC 174 FINAL EXAMINATION - APRIL 2018
INSTRUCTORS: MANSOURI, GHARESIFARD, YUI

INSTRUCTIONS

- This examination is **3 hours** in length and consists of **6 questions**.
- **READ THE QUESTIONS CAREFULLY!**
- Answer **all questions**, writing **clearly** in the space provided.
- If you need more room, there are blank pages at the end of the test. **If you use these pages, you must provide clear directions to the marker**, e.g. continued on page 20.
- SHOW ALL YOUR WORK, **clearly and in order**, if you wish to receive full credit.
- **No textbook, lecture note, calculator, computer, or other aid of any sort is allowed**.
- PLEASE NOTE: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer the exam questions as written.
- Good luck!

Q1	Q2	Q3	Q4	Q5	Q6	Total
20	10	15	20	20	15	100

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Problem 1

Consider the real 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) **Determine the set of all eigenvalues of** A. [5 pts]

The characteristic polynomial of A is given by

$$p(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 1 \end{pmatrix} = (\lambda - 2)^2(\lambda - 1)$$

Hence, the set of eigenvalues of A is $\{1, 2\}$.

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(Problem 1 - Cont'd)

(b) **Determine whether or not A is invertible.** [5 pts]

We have that $\det(A) = 2 \times 2 \times 1 = 4 \neq 0$, and hence A is invertible. Alternatively, A does not have any zero eigenvalue and hence is invertible.

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(Problem 1 - Cont'd)

(c) **Determine whether or not A is diagonalizable.** `[10 pts]`

Recall that A is diagonalizable if and only if there exists a basis of $\widehat{\mathbb{R}^3}$ (since A is 3×3) made of eigenvectors of A. We should therefore identify the eigenspaces of A.

- The eigenspace of A corresponding to eigenvalue 2 is given by $\ker(2I-A)$; we have, $\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(2I-A) \;&\Leftrightarrow\; \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow\; \begin{pmatrix} -y \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

  Noting that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \ker(2I-A)$, this shows that $(\mathbf{v}_1)$ is a generating family for $\ker(2I-A)$. Furthermore, since $\mathbf{v}_1 \neq \mathbf{0}$, it follows that $(\mathbf{v}_1)$ is a basis for $\ker(2I-A)$.

- The eigenspace of A corresponding to eigenvalue 1 is given by $\ker(I-A)$; we have, $\forall \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(I-A) \;&\Leftrightarrow\; \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow\; \begin{pmatrix} x-y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow\; x-y=y=0 \\
&\Leftrightarrow\; x=y=0 \\
&\Leftrightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \\
&\Leftrightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

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Noting that $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \ker(I - A)$, this shows that $(\mathbf{v}_2)$ is a generating family for $\ker(I - A)$. Furthermore, since $\mathbf{v}_2 \neq \mathbf{0}$, it follows that $(\mathbf{v}_2)$ is a basis for $\ker(I - A)$.

If we had a basis $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ of $\widehat{\mathbb{R}^3}$ with $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ eigenvectors of A, then at least two of them would be elements of the same eigenspace, since, as we saw above, A has only two distinct eigenvalues and hence two distinct eigenspaces. Now if two of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ were to be elements of the eigenspace $\ker(2I - A)$, then they would both be non-zero multiples of $\mathbf{v}_1$, and hence linearly dependent. Hence, this is not possible (since we assumed $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ was a basis of $\widehat{\mathbb{R}^3}$). Similarly, if they were both elements of $\ker(I - A)$, they would both be non-zero multiples of $\mathbf{v}_2$, and hence again linearly dependent, and hence this is not possible either. We conclude: There is no basis of $\widehat{\mathbb{R}^3}$ made of eigenvectors of A. Hence A is not diagonalizable.

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Problem 2

Let A be a real 2×2 matrix, and assume A has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$. **Determine whether or not A is diagonalizable**. [10 pts]

Since λ_1 is an eigenvalue of A, there exists $\mathbf{v}_1 \neq \mathbf{0}$ such that $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. Similarly, Since λ_2 is an eigenvalue of A, there exists $\mathbf{v}_2 \neq \mathbf{0}$ such that $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$. We assume $\lambda_1 \neq \lambda_2$. Let us now show that $(\mathbf{v}_1, \mathbf{v}_2)$ forms a linearly independent family of $\widehat{\mathbb{R}^2}$ (and hence a basis of $\widehat{\mathbb{R}^2}$, since the latter has dimension 2).

Suppose to the contrary that $(\mathbf{v}_1, \mathbf{v}_2)$ forms a linearly dependent family of $\widehat{\mathbb{R}^2}$; then, one of $\mathbf{v}_1, \mathbf{v}_2$ is a linear combination of the other, i.e. a scalar multiple of the other; note that this scalar is necessarily non-zero since $\mathbf{v}_1, \mathbf{v}_2$ are assumed to be eigenvectors of A (and hence are distinct from the zero vector). Hence, there would exist some $\alpha \in \mathbb{R}$, with $\alpha \neq 0$, such that $\mathbf{v}_1 = \alpha\mathbf{v}_2$. But then we would obtain

$$A\mathbf{v}_1 = A(\alpha\mathbf{v}_2) = \alpha A\mathbf{v}_2,$$

i.e.

$$\lambda_1\mathbf{v}_1 = \alpha\lambda_2\mathbf{v}_2 = \lambda_2(\alpha\mathbf{v}_2) = \lambda_2\mathbf{v}_1,$$

which then yields

$$(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0},$$

and since $\mathbf{v}_1 \neq \mathbf{0}$, it follows that $\lambda_1 - \lambda_2 = 0$, contradicting our assumption that $\lambda_1 \neq \lambda_2$. It follows therefore that $(\mathbf{v}_1, \mathbf{v}_2)$ forms a linearly independent family of $\widehat{\mathbb{R}^2}$, and since $\widehat{\mathbb{R}^2}$ has dimension 2, it follows that $(\mathbf{v}_1, \mathbf{v}_2)$ forms a basis of $\widehat{\mathbb{R}^2}$. Hence A is diagonalizable.

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Problem 3

Let $(\mathbf{V}, +, \cdot)$ be a real vector space.

(a) Assume first $\mathbf{V}$ has dimension 3, and let $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be a basis for $\mathbf{V}$. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbf{V}$ be defined by $\mathbf{w}_1 = \mathbf{v}_1$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. **Determine whether or not $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is a basis for $\mathbf{V}$.** `[5 pts]`

We first check for independence: suppose

$$\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \alpha_3 \mathbf{w}_3 = 0,$$

for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Then

$$\begin{aligned}
&\alpha_1 \mathbf{v}_1 + \alpha_2(\mathbf{v}_1 + \mathbf{v}_2) + \alpha_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 0 \\
\Rightarrow &(\alpha_1 + \alpha_2 + \alpha_3)\mathbf{v}_1 + (\alpha_2 + \alpha_3)\mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = 0 \\
\Rightarrow &\begin{cases} (\alpha_1 + \alpha_2 + \alpha_3) = 0 \\ (\alpha_2 + \alpha_3) = 0 \\ \alpha_3 = 0 \end{cases} \qquad \text{(by independence of } (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3))
\end{aligned}$$

This clearly implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which yields the independence of the vectors in $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. We now check whether or not $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is a generating set: let $\mathbf{v} \in V$ be any vector. Then we wish to know if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \alpha_3 \mathbf{w}_3$$

By the calculation in the previous part this means that

$$\mathbf{v} = (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{v}_1 + (\alpha_2 + \alpha_3)\mathbf{v}_2 + \alpha_3 \mathbf{v}_3$$

However, since $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a generating set for V, there exist $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \beta_3 \mathbf{v}_3$$

Hence, by choosing

$$\begin{aligned}
\alpha_3 &= \beta_3 \\
\alpha_2 &= \beta_2 - \beta_3 \\
\alpha_1 &= \beta_1 - \beta_2 - \beta_3
\end{aligned}$$

we have $\mathbf{v} = \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \alpha_3 \mathbf{w}_3$, i.e., $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is a generating set. Therefore, $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is a basis.

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(Problem 3 - Cont'd)

(b) Let now $\mathbf{v}_1, \mathbf{v}_2$ be linearly independent vectors of $\mathbf{V}$, and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{V}$ be defined by $\mathbf{u}_1 = \mathbf{v}_1, \mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2, \mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2$. Let $\mathbf{U}$ denote the linear span of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. **Compute the dimension of U.** `[5 pts]`

First, note that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are in the span of $\mathbf{v}_1, \mathbf{v}_2$, and hence the dimension of $\mathbf{U}$ is at most 2, i.e,. $\dim(\mathbf{U}) \leq 2$. We claim that $\mathbf{u}_1$ and $\mathbf{u}_2$ are independent: let

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 = \mathbf{0}$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$. Hence,

$$\alpha_1 \mathbf{v}_1 + \alpha_2 (\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0} \quad \Rightarrow (\alpha_1 + \alpha_2)\mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \mathbf{0}.$$

thus, $\alpha_1 = \alpha_2 = 0$, i.e., $\mathbf{u}_1$ and $\mathbf{u}_2$ are independent as claimed. This means that $\dim(\mathbf{U}) \geq 2$, which along with the observation made above yields that $\dim(\mathbf{U}) = 2$.

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(Problem 3 - Cont'd)

(c) Consider now the real vector space $\widehat{\mathbb{R}^3}$, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \widehat{\mathbb{R}^3}$ be defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ k \\ k^2 \end{pmatrix},$$

where $k \in \mathbb{R}$. **Determine for which values of k we have that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ forms a basis for $\widehat{\mathbb{R}^3}$.** `[5 pts]`

First, since $\widehat{\mathbb{R}^3}$ is of dimension 3, any three independent vectors in $\widehat{\mathbb{R}^3}$ forms a basis. Therefore, it is enough to seek for values of $k \in \mathbb{R}$ that ensure the independence of $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Let now

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0},$$

$\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, i.e.,

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ k \\ k^2 \end{pmatrix} = \mathbf{0},$$
$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To guarantee independence, we wish to have no solution except for $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Said differently, we wish for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{pmatrix}$$

to have a zero kernel, i.e., A has a non-zero determinant. We now compute the determinant of A

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{vmatrix} = 1 \begin{vmatrix} -1 & k \\ 1 & k^2 \end{vmatrix} - 11 \begin{vmatrix} 1 & k \\ 1 & k^2 \end{vmatrix} + 11 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$
$$= (-k^2 - k) - (k^2 - k) + (1 + 1) = -2k^2 + 2$$

Hence,

$$\det(A) \neq 0 \quad \iff \quad k \neq \pm 1,$$

which means $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ forms a basis for $\widehat{\mathbb{R}^3}$ if and only if $k \neq \pm 1$.

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Problem 4

Consider the system of linear equations given by:

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 1, \\ x_1 + 2x_2 + 3x_3 &= 4, \\ ax_1 + ax_2 + 2ax_3 + x_4 &= 5, \end{aligned}$$

where we wish to solve for the quadruple (x_1, x_2, x_3, x_4) of real numbers, and where a is a real parameter.

(a) **Write the augmented matrix for this system.** [5 pts]

It is given by

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 0 & 4 \\ a & a & 2a & 1 & 5 \end{pmatrix}$$

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(Problem 4 - Cont'd)

(b) **Transform the augmented matrix to row-echelon form using a sequence of elementary row operations (clearly indicate which elementary row operation you perform at each step).** [5 pts]

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 0 & 4 \\ a & a & 2a & 1 & 5 \end{pmatrix}$$

$$(R2 - R1 \to R2) \Rightarrow \quad \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ a & a & 2a & 1 & 5 \end{pmatrix}$$

$$(R3 - aR1 \to R3) \Rightarrow \quad \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1-a & 5-a \end{pmatrix}$$

Hence,

- if $a \neq 1$

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & \dfrac{5-a}{1-a} \end{pmatrix}$$

- if $a = 1$

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5-a \end{pmatrix}$$

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(Problem 4 - Cont'd)

(c) **Using (b), determine all the values of a for which the system has no solution**. [5 pts]

Clearly, the the system has no solution if and only if $a = 1$.

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(Problem 4 - Cont'd)

(d) **Let now $a = 5$; determine the set of all solutions of the system using back-substitution.** [5 pts]

For $a = 5$, the augmented matrix to row-echelon is given by

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Hence, by back-substitution, we have that

$$\begin{aligned} x_4 &= 0 \\ x_2 + x_3 - x_4 &= 3 \\ x_1 + x_2 + 2x_3 + x_4 &= 1 \end{aligned}$$

which, up on simplification, gives

$$\begin{aligned} x_4 &= 0 \\ x_2 &= 3 - x_3 \\ x_1 &= -2 - x_3, \end{aligned}$$

with $x_3 \in \mathbb{R}$. Hence, the system has infinitely many solutions given by the set S

$$\begin{aligned} S &= \{(2 - x_3, 3 - x_3, x_3, 0) \in \mathbb{R}^4 \mid x_3 \in \mathbb{R}\} \\ &= \{(0, 3, 0, 0) + x_3(2, -1, 1, 0) \in \mathbb{R}^4 \mid x_3 \in \mathbb{R}\}. \end{aligned}$$

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Problem 5

Consider the real 3×3 matrices A and B given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & a+1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

where a is a real parameter.

(a) **Compute AB and BA.** `[5 pts]`

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 2 & a+1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 3a+7 & 2a+4 & 2a \\ 0 & 0 & -2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & a+1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a+2 & -1 \\ 9 & 2a & 2 \\ 4 & a & 1 \end{pmatrix}$$

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(Problem 5 - Cont'd)

(b) **Compute the determinant** $\det(A)$ **of** A **and determine all the values of** a **for which** A **is invertible**. `[5 pts]`

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & a+1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} a+1 & 0 \\ -1 & 1 \end{vmatrix} = (a+1).$$

Hence, A is invertible if and only if $a \neq -1$.

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(Problem 5 - Cont'd)

(c) **Determine whether or not** B **is invertible by computing the determinant** $\det(B)$ **of** B. `[5 pts]`

$$\det(B) = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 0 - 1 - 1 = -2 \neq 0$$

Hence, B is invertible.

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(Problem 5 - Cont'd)

(d) **Compute** $\det(AABBAB)$. `[5 pts]`

$$\det(AABBAB) = \det(A)^3 \det(B)^3 = (a+1)^3(-2)^3 = -8(a+1)^3$$

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Problem 6

Let A be the real 3×4 matrix given by

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 3 & 0 & 6 & -3 \\ -2 & 0 & -4 & 2 \end{pmatrix}.$$

(a) **Find a basis for** $\mathrm{Ker}(A)$ **and compute the dimension of** $\mathrm{Ker}(A)$. `[5 pts]`

Any vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ that is in the kernel of A satisfies

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 3 & 0 & 6 & -3 \\ -2 & 0 & -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To find a basis for the kernel, we solve the corresponding system of linear equations by forming the augmented matrix and computing the row-echelon form:

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 3 & 0 & 6 & -3 & 0 \\ -2 & 0 & -4 & 2 & 0 \end{pmatrix} \\
(R2/3 \to R2) \Rightarrow \quad &\begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ -2 & 0 & -4 & 2 & 0 \end{pmatrix} \\
(R3/(-2) \to R3) \Rightarrow \quad &\begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 0 & 2 & -1 & 0 \end{pmatrix} \\
(R3 - R2 \to R3) \Rightarrow \quad &\begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
(R1 - R2 \to R2) \Rightarrow \quad &\begin{pmatrix} 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

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Thus if $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathrm{Ker}(A)$, then $x_2 = 0$ and $x_1 + x_2 + 2x_3 - x_4 = 0$. Hence,

$$\mathrm{Ker}(A) = \{x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mid x_3, x_4 \in \mathbb{R}\}.$$

Since $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are easily seen to be independent, $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for $\mathrm{Ker}(A)$; hence the dimension of $\mathrm{Ker}(A)$ is 2.

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(Problem 6 - Cont'd)

(b) **Find a basis for** $\mathrm{Im}(A)$ **and compute the dimension of** $\mathrm{Im}(A)$. `[5 pts]`

$$\mathrm{Im}(A) = \mathrm{Span}\{\begin{pmatrix}1\\3\\-2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\6\\-4\end{pmatrix}, \begin{pmatrix}-1\\-3\\2\end{pmatrix}\}$$

The last two vectors are easily seen to be multiple factors of the first vector, and hence

$$\mathrm{Im}(A) = \mathrm{Span}\{\begin{pmatrix}1\\3\\-2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\}.$$

These two vectors are easily seen to be linearly independent and hence the dimension of $\mathrm{Im}(A)$ is 2.

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(Problem 6 - Cont'd)

(c) **Verify the rank-nullity theorem using the dimensions computed in (a) and (b).** `[5 pts]`
Since $\dim(\mathrm{Im}(A)) = \dim(\mathrm{Ker}(A)) = 2$, the rank-nullity theorem is verified:

$$\dim(\mathrm{Im}(A)) + \dim(\mathrm{Ker}(A)) = \dim(\mathbb{R}^4) = 4.$$

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Space for additional work. **Indicate clearly which question you are continuing if you use this space.**

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