HAND IN
answers recorded
on exam paper

STUDENT NUMBER: ______________________

APSC 174 — Final Exam

Faculty of Arts and Science

Monday, April 15, 2019

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INSTRUCTIONS: The exam has eight questions, worth a total of 100 marks.

The exam is three hours in length.

Answer **all questions**, writing clearly in the space provided. If you need more room, continue to answer on the back of the **previous page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order.

Calculators, data sheets, or other aids are *not* permitted.

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer the questions as written.

Student ID number (please write as legibly as possible within the boxes)

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	Total
/10	/12	/12	/8	/18	/14	/10	/16	

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[10 pts] **1.** Consider $C^\infty(\mathbb{R})$, the vector space of all functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ having derivatives of arbitrary order. Recall that $C^\infty(\mathbb{R})$ is a vector space under the usual addition and scalar multiplication of functions.

Let $\mathbf{W} = \{f \in C^\infty(\mathbb{R}) \; : \; f'(x) = 2x \cdot f(x)\} \subset C^\infty(\mathbb{R})$.

For instance, $e^{x^2} \in \mathbf{W}$ since (by the chain rule) if $f(x) = e^{x^2}$,

$$f'(x) = (x^2)' \cdot e^{x^2} = 2x \cdot e^{x^2} = 2x \cdot f(x).$$

On the other hand, $\sin(x) \notin \mathbf{W}$ since if $f(x) = \sin(x)$,

$$f'(x) = \cos(x) \neq 2x \cdot \sin(x) = 2x \cdot f(x).$$

Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

Solution. We check the conditions to be a subspace.

(i) **Q:** Is the zero vector of $C^\infty(\mathbb{R})$ in $\mathbf{W}$?

A: Yes. The zero vector $\mathbf{0}$ of $C^\infty(\mathbb{R})$ is the constant zero function $f(x) = 0$ for all $x \in \mathbb{R}$. For this f, we have $f'(x) = 0$ and $2xf(x) = 0$ for all $x \in \mathbb{R}$, so $f'(x) = 2xf(x)$. Thus $\mathbf{0} \in \mathbf{W}$.

(ii) **Q:** If $f_1, f_2 \in \mathbf{W}$, is $f_1 + f_2 \in \mathbf{W}$?

A: Yes. Since $f_1 \in \mathbf{W}$, we know that $f_1'(x) = 2xf_1(x)$. Similarly, since $f_2 \in \mathbf{W}$, we know that $f_2'(x) = 2xf_2(x)$. Thus

$$(f_1 + f_2)'(x) = f_1'(x) + f_2'(x) = 2xf_1(x) + 2xf_2(x) = 2x\big(f_1(x) + f_2(x)\big) = 2x(f_1 + f_2)(x),$$

which shows that $f_1 + f_2 \in \mathbf{W}$.

(iii) **Q:** If $\alpha \in \mathbb{R}$ and $f \in \mathbf{W}$, is $\alpha f \in \mathbf{W}$?

A: Yes. Since $f \in \mathbf{W}$, we know that $f'(x) = 2xf(x)$. Then

$$(\alpha f)'(x) = \alpha f'(x) = \alpha\big(2xf(x)\big) = 2x(\alpha f(x)) = 2x(\alpha f)(x)$$

and so $\alpha f \in \mathbf{W}$.

Since $\mathbf{W}$ passes all three tests, $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

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2.

[6 pts] (a) Use row operations to put the matrix below into Row Reduced Echelon Form (RREF).

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 2 & -1 & -7 & 9 \\ 1 & 2 & 4 & 2 \end{bmatrix}$$

Solution. The RREF of the matrix is

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A possible sequence of row operations for putting the matrix into RREF is

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 2 & -1 & -7 & 9 \\ 1 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R2}-2\text{R1}\mapsto\text{R2}} \begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & -7 & -21 & 7 \\ 1 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R3}-\text{R1}\mapsto\text{R3}} \begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & -7 & -21 & 7 \\ 0 & -1 & -3 & 1 \end{bmatrix} \xrightarrow{(-1/7)\text{R2}\mapsto\text{R2}}$$

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & -1 & -3 & 1 \end{bmatrix} \xrightarrow{\text{R3}+\text{R2}\mapsto\text{R3}} \begin{bmatrix} 1 & 3 & 7 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{R1}-3\text{R2}\mapsto\text{R1}} \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Or

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 2 & -1 & -7 & 9 \\ 1 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R1} - \text{R3} \mapsto \text{R1}} \begin{bmatrix} 0 & 1 & 3 & -1 \\ 2 & -1 & -7 & 9 \\ 1 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{R3} - 2\text{ R1} \mapsto \text{R3}} \begin{bmatrix} 0 & 1 & 3 & -1 \\ 2 & -1 & -7 & 9 \\ 1 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{R2} + \text{R1} \mapsto \text{R2}}$$

$$\begin{bmatrix} 0 & 1 & 3 & -1 \\ 2 & 0 & -4 & 8 \\ 1 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{R2} - 2\text{ R3} \mapsto \text{R2}} \begin{bmatrix} 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{Reorder}} \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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[4 pts] (b) Parametrize all the solutions to the system of linear equations below. Write your answer in vector form. [NOTE : Part (a) is relevant.]

$$\begin{array}{rcrcrcr} x & + & 3y & + & 7z & = & 1 \\ 2x & - & y & - & 7z & = & 9 \\ x & + & 2y & + & 4z & = & 2 \end{array}$$

Solution. The augmented matrix corresponding to the system is

$$\left[\begin{array}{rrr|r} 1 & 3 & 7 & 1 \\ 2 & -1 & -7 & 9 \\ 1 & 2 & 4 & 2 \end{array}\right]$$

which is exactly the matrix from part (a). Writing down the RREF obtained in part (a) with the corresponding variables, we get

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{rrr|r} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]. \end{array}$$

The dependent variables are x and y; there is one free variable, z. Setting $z = t$, we get the equations

$$\begin{array}{rcr} x - 2z = 4 & \text{or} & x = 2z + 4 = 2t + 4 \\ y + 3z = -1 & \text{or} & y = -3z - 1 = -3t - 1 \end{array}$$

In vector form this is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t+4 \\ -3t-1 \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

[2 pts] (c) Find α_1, α_2, $\alpha_3 \in \mathbb{R}$ so that $(1, 9, 2) = \alpha_1(1, 2, 1) + \alpha_2(3, -1, 2) + \alpha_3(7, -7, 4)$.

Solution. The scalars α_1, α_2, α_3 solve the vector equation $(1, 9, 2) = \alpha_1(1, 2, 1) + \alpha_2(3, -1, 2) + \alpha_3(7, -7, 4)$ precisely when, after renaming them as $\alpha_1 = x$, $\alpha_2 = y$, and $\alpha_3 = z$, they solve the system of linear equations in part (b). For example, setting the free parameter t to zero in the parametrized solution obtained in part (b), we get

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

i.e., $\alpha_1 = 4$, $\alpha_2 = -1$, and $\alpha_3 = 0$.

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3. Suppose that $L\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and we know that $L(5,2)=(3,2,-4)$ and that $L(2,1)=(1,2,-1)$.

[8 pts] (a) Find the standard matrix for L.

Solution. The standard matrix of L has column vectors $L(1,0)$ and $L(0,1)$. We can obtain these vectors by writing up $(1,0)$ and $(0,1)$ as linear combinations of $(5,2)$ and $(2,1)$ and using the linearity of L.

Finding the linear combinations: We have to find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1(5,2)+\alpha_2(2,1)=(1,0) \quad \text{and} \quad \beta_1(5,2)+\beta_2(2,1)=(0,1),$$

i.e., we have to solve the two systems of linear equations

$$\begin{array}{rcl} 5\alpha_1+2\alpha_2 &=& 1 \\ 2\alpha_1+\alpha_2 &=& 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} 5\beta_1+2\beta_2 &=& 0 \\ 2\beta_1+\beta_2 &=& 1. \end{array}$$

The second equation of the first system gives $\alpha_2=-2\alpha_1$. Plugging this into the first equation we get $5\alpha_1-4\alpha_1=1$, i.e., $\underline{\alpha_1=1}$, which then gives $\underline{\alpha_2=-2}$.
Similarly, the first equation of the second system gives $\beta_2=-\frac{5}{2}\beta_1$. Plugging this into the second equation gives $2\beta_1-\frac{5}{2}\beta_1=1$, i.e., $\underline{\beta_1=-2}$, which then gives $\underline{\beta_2=5}$. Thus

$$(1,0)=(5,2)-2\cdot(2,1) \quad \text{and} \quad (0,1)=-2\cdot(5,2)+5\cdot(2,1).$$

Finding the standard matrix: Since we know that $L(5,2)=(3,2,-4)$ and $L(2,1)=(1,2,-1)$, we can use the linearity of L to compute $L(1,0)$ and $L(0,1)$ as

$$\begin{aligned} L(1,0) &= L\big((5,2)-2\cdot(2,1)\big)=L(5,2)-2L(2,1) \\ &= (3,2,-4)-2\cdot(1,2,-1) \\ &= (1,-2,-2) \end{aligned}$$

and

$$\begin{aligned} L(0,1) &= L\big(-2\cdot(5,2)+5\cdot(2,1)\big)=-2L(5,2)+5L(2,1) \\ &= -2\cdot(3,2,-4)+5\cdot(1,2,-1) \\ &= (-1,6,3). \end{aligned}$$

Thus the standard matrix for L is

$$\begin{bmatrix} 1 & -1 \\ -2 & 6 \\ -2 & 3 \end{bmatrix}.$$

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Alternate Solution. Let A be the standard matrix for L, with entries unknowns :

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

We are given the conditions that

$$(*) \qquad \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},$$

or, combining both equations, that

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ -4 & -1 \end{bmatrix}.$$

The most efficient way to solve these equations is to multiply on the right by the inverse of the 2×2 matrix, to get

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 6 \\ -2 & 3 \end{bmatrix}.$$

Alternatively, $(*)$ can be written as three separate systems of linear equations

$$\begin{array}{rcrcr} 5a & + & 2b & = & 3 \\ 2a & + & b & = & 1 \end{array}, \quad \begin{array}{rcrcr} 5c & + & 2d & = & 2 \\ 2c & + & d & = & 2 \end{array}, \quad \text{and} \quad \begin{array}{rcrcr} 5e & + & 2f & = & -4 \\ 2e & + & f & = & -1 \end{array},$$

which can be solved individually to give $a = 1$, $b = -1$, $c = -2$, $d = 6$, $e = -2$, and $f = 3$, giving the standard matrix as before.

[4 pts] (b) Find $L(4, 1)$.

Solution. We use the linearity of L and part (a) to calculate

$$\begin{aligned} L(4,1) &= L\big(4(1,0) + (0,1)\big) = 4L(1,0) + L(0,1) \\ &= 4(1,-2,-2) + (-1,6,3) \\ &= (3,-2,-5) \end{aligned}$$

so that $\underline{L(4,1) = (3,-2,-5)}$.

Alternate Solution.

$$L(4,1) = \begin{bmatrix} 1 & -1 \\ -2 & 6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}.$$

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4. The matrix A and its RREF are shown below.

$$A = \begin{bmatrix} 2 & 1 & -4 & 3 & 9 \\ 1 & 5 & 7 & 2 & 1 \\ 3 & 0 & -9 & 0 & 6 \\ 1 & 1 & -1 & 0 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $L\colon \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ be the linear transformation whose standard matrix is A.

[4 pts] (a) Find a basis for $\operatorname{Im}(L)$.

Solution. Since the leading ones appear in the first, second, and fourth columns of the RREF, by our algorithm from class, the first, the second, and the fourth columns of A, the standard matrix for L, form a basis for $\operatorname{Im}(L)$. Thus

$$\big((2,1,3,1),\ (1,5,0,1),\ (3,2,0,0)\big)$$

is a basis for $\operatorname{Im}(L)$.

[4 pts] (b) Find a basis for $\operatorname{Ker}(L)$.

Solution. Adding an extra column of zeros to the RREF we get the matrix

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

We will use the algorithm learned in class the parametrize all solutions to the corresponding system of linear equations. The dependent variables x_1, x_2 and x_4 correspond to the columns with the leading ones. The independent (free) variables are $x_3 = t_1$ and $x_5 = t_2$. The augmented RREF gives the equations

$$x_1 - 3t_1 + 2t_2 = 0, \quad x_2 + 2t_1 - t_2 = 0, \quad x_4 + 2t_2 = 0,$$

i.e.,

$$x_1 = 3t_1 - 2t_2, \quad x_2 = -2t_1 + t_2, \quad x_4 = -2t_2.$$

Thus all solutions are parametrized as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3t_1 - 2t_2 \\ -2t_1 + t_2 \\ t_1 \\ -2t_2 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

By our algorithm for finding a basis for $\operatorname{Ker}(L)$, the list

$$\big((3,-2,1,0,0),(-2,1,0,-2,1)\big)$$

is a basis for $\operatorname{Ker}(L)$.

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5. Solve the system of linear equations below :

$$\begin{array}{rcrcrcr} 2x & + & y & + & 3z & = & 4 \\ x & & & + & z & = & 1 \\ & & 2y & + & 3z & = & 8 \end{array}$$

Solution. Encoding the system as an augmented matrix and putting it into RREF we get

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 3 & 8 \end{array}\right] \overset{\text{RREF}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Thus the unique solution is $x = -3$, $y = -2$, and $z = 4$.

A possible sequence of row operations for putting the matrix into RREF is

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 3 & 8 \end{array}\right] \overset{\text{R1}-2\text{ R2}\mapsto\text{R1}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 3 & 8 \end{array}\right] \overset{\text{R3}-2\text{ R1}\mapsto\text{R3}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right] \overset{\text{R1}-\text{R3}\mapsto\text{R1}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\overset{\text{R2}-\text{R3}\mapsto\text{R2}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right] \overset{\text{Reorder}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

[5 pts] (b) Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$. Compute A^{-1}.

Solution. Writing down the matrix $\left[\begin{array}{c|c} A & I_3 \end{array}\right]$ and putting this matrix into RREF we get

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \overset{\text{RREF}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -3 & -1 \\ 0 & 1 & 0 & 3 & -6 & -1 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right],$$

so that $A^{-1} = \begin{bmatrix} 2 & -3 & -1 \\ 3 & -6 & -1 \\ -2 & 4 & 1 \end{bmatrix}$.

Any sequence of row operations which worked in part (a) also works in (b). For instance,

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \overset{\text{R1}-2\text{ R2}\mapsto\text{R1}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \overset{\text{R3}-2\text{ R1}\mapsto\text{R3}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right] \overset{\text{R1}-\text{R3}\mapsto\text{R1}}{\rightsquigarrow}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 3 & -6 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right] \overset{\text{R2}-\text{R3}\mapsto\text{R2}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 3 & -6 & -1 \\ 1 & 0 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right] \overset{\text{Reorder}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & -1 \\ 0 & 1 & 0 & 3 & -6 & -1 \\ 0 & 0 & 1 & -2 & 4 & 1 \end{array}\right].$$

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(— problem 5 continued —)

[4 pts] (c) Let $\mathbf{w} = (4, 1, 8)$. Compute $A^{-1}\mathbf{w}$.

Solution.

$$A^{-1}\mathbf{w} = \begin{bmatrix} 2 & -3 & -1 \\ 3 & -6 & -1 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ -6 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix}.$$

[4 pts] (d) Explain the connection between your answers in (a) and (c).

Solution. The system of equations from (a), in matrix form, is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \quad \text{or } A\mathbf{v} = \mathbf{w},$$

with $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (4, 1, 8)$. Multiplying both sides of this equation by A^{-1} we get

$$A^{-1}\mathbf{w} = A^{-1}A\mathbf{v} = (A^{-1}A)\mathbf{v} = I_3\mathbf{v} = \mathbf{v},$$

or $(x, y, z) = A^{-1}(4, 1, 8)$.

Alternate Solution. As above, we are trying to solve the matrix equation $A\mathbf{v} = \mathbf{w}$, with $\mathbf{w} = (4, 1, 8)$. Viewing A as a linear map $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, we are looking for a vector $\mathbf{v}$ sent to $(4, 1, 8)$ by this map. The inverse matrix A^{-1} undoes the linear map given by A, and so $\mathbf{v} = A^{-1}(4, 1, 8)$ is the solution we are looking for.

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6. Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

[6 pts] (a) Compute $\det(A)$.

Solution. Using, e.g., Laplace expansion by the third row we get

$$\begin{aligned} \det \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} &= 2\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - 0\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} + 3\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \\ &= 2(0-1) + 0(0-1) + 3(3-1) = 4. \end{aligned}$$

[6 pts] (b) Compute $\det(B)$.

Solution. Using, e.g., Laplace expansion by the second row we get

$$\det \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} = -2(4-1) + 0(6-2) - 1(3-4) = -5.$$

[2 pts] (c) Compute $\det(AB)$.

Solution. For any two square matrices A and B of the same size we have

$$\det(AB) = \det(A)\det(B).$$

Thus $\underline{\det(AB) = 4(-5) = -20}$.

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7. Let $L\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

[2 pts] (a) State the rank-nullity theorem for L.

Solution.

$$\dim(\mathrm{Im}(L)) + \dim(\mathrm{Ker}(L)) = n.$$

[2 pts] (b) State, without proof, what "L is surjective" means in terms of the dimension of $\mathrm{Im}(L)$.

Solution. L is surjective if and only if $\dim(\mathrm{Im}(L)) = m$.

[2 pts] (c) State, without proof, what "L is injective" means in terms of the dimension of $\mathrm{Ker}(L)$.

Solution. L is injective if and only if $\dim(\mathrm{Ker}(L)) = 0$.

[4 pts] (d) Suppose we know that L is surjective. State, with proof, which inequality must hold between n and m. (I.e., which is bigger, and why.)

Solution. Since L is surjective, by (b) we have $\dim(\mathrm{Im}(L)) = m$. From (a) we have

$$n = \dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(\mathrm{Ker}(L)) + m.$$

Since $\dim(\mathrm{Ker}(L)) \geqslant 0$, we conclude that $n \geqslant m$.

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8. Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$.

[6 pts] (a) Verify that $\mathbf{v}_1 = (3, 2)$ and $\mathbf{v}_2 = (1, -1)$ are eigenvectors of A and find their eigenvalues.

Solution. For $\mathbf{v}_1$ we have

$$A\mathbf{v}_1 = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{v}_1.$$

Thus $\mathbf{v}_1$ is an eigenvector of A of eigenvalue $\lambda_1 = 1$. For $\mathbf{v}_2$ we have

$$A\mathbf{v}_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} - 1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{6}\mathbf{v}_2.$$

Thus $\mathbf{v}_2$ is also an eigenvector of A of eigenvalue $\lambda_2 = \frac{1}{6}$.

[2 pts] (b) Find $A^2\mathbf{v}_1$ and $A^2\mathbf{v}_2$.

Solution. If $\mathbf{v}$ is an eigenvector of A of eigenvalue λ, then $A^n\mathbf{v} = \lambda^n\mathbf{v}$ for any integer $n \geqslant 1$. Thus

$$A^2\mathbf{v}_1 = 1^2\mathbf{v}_1 = \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and

$$A^2\mathbf{v}_2 = \left(\tfrac{1}{6}\right)^2\mathbf{v}_2 = \begin{bmatrix} \frac{1}{36} \\ -\frac{1}{36} \end{bmatrix}.$$

[4 pts] (c) Let $\mathbf{w} = (9, 1)$. Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution. We have to find $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha(3, 2) + \beta(1, -1) = (9, 1)$$

i.e., we have to solve the two systems of linear equations

$$\begin{aligned} 3\alpha + \beta &= 9 \\ 2\alpha - \beta &= 1. \end{aligned}$$

One can solve this system by, e.g., encoding it in a matrix and putting the matrix into RREF:

$$\left[\begin{array}{cc|c} 3 & 1 & 9 \\ 2 & -1 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

from which we see that $\alpha = 2$ and $\beta = 3$ is the unique solution; i.e., that

$$\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2.$$

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[2 pts] (d) For $n \geqslant 1$, find a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of A.

Solution. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A of eigenvalues 1 and $\frac{1}{6}$, respectively, we know that $A^n\mathbf{v}_1 = 1^n\mathbf{v}_1 = \mathbf{v}_1$ and $A^n\mathbf{v}_2 = \left(\frac{1}{6}\right)^n\mathbf{v}_2$. Using part (c) and the linearity of A, we obtain

$$\begin{aligned} A^n\mathbf{w} &= A^n\big(2\mathbf{v}_1 + 3\mathbf{v}_2\big) = 2A^n\mathbf{v}_1 + 3A^n\mathbf{v}_2 \\ &= 2\mathbf{v}_1 + 3\big(\tfrac{1}{6}\big)^n\mathbf{v}_2 \\ &= 2\cdot(3,2) + 3\cdot\big(\tfrac{1}{6}\big)^n(1,-1) \\ &= \Big(6 + 3\big(\tfrac{1}{6}\big)^n,\ 4 - 3\big(\tfrac{1}{6}\big)^n\Big). \end{aligned}$$

[2 pts] (e) Find $\lim\limits_{n\to\infty} A^n\mathbf{w}$.

Solution. Since $\lim\limits_{n\to\infty}\left(\frac{1}{6}\right)^n = 0$, we have

$$\lim_{n\to\infty} A^n\mathbf{w} = \lim_{n\to\infty}\Big(6 + 3\big(\tfrac{1}{6}\big)^n,\ 4 - 3\big(\tfrac{1}{6}\big)^n\Big) = (6,4).$$

Alternate Solution. Using the answer from (d) we obtain

$$\lim_{n\to\infty} A^n\mathbf{w} = \lim_{n\to\infty}\left(2\mathbf{v}_1 + 3\left(\tfrac{1}{6}\right)^n\mathbf{v}_2\right) = 2\mathbf{v}_1 + 3\cdot 0\cdot\mathbf{v}_2 = 2\mathbf{v}_1 = (6,4).$$