HAND IN
answers recorded
on exam paper

STUDENT NUMBER:

# APSC 174 — Final Exam

Faculty of Arts and Science

Monday, April 15, 2019

T. Linder M. Roth S. Yüksel

Instructions: The exam has eight questions, worth a total of 100 marks.

The exam is three hours in length.

Answer **all questions**, writing clearly in the space provided. If you need more room, continue to answer on the back of the **previous page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order.

Calculators, data sheets, or other aids are *not* permitted.

**Please Note:** Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer the questions as written.

Student ID number (please write as legibly as possible within the boxes)

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | |



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|
| /10 | /12 | /12 | /8 | /18 | /14 | /10 | /16 | |

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER:

[10 pts] **1.** Consider $C^\infty(\mathbb{R})$, the vector space of all functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ having derivatives of arbitrary order. Recall that $C^\infty(\mathbb{R})$ is a vector space under the usual addition and scalar multiplication of functions.

Let $\mathbf{W} = \left\{ f \in C^\infty(\mathbb{R}) \ : \ f'(x) = 2x \cdot f(x) \right\} \subset C^\infty(\mathbb{R})$.

For instance, $e^{x^2} \in \mathbf{W}$ since (by the chain rule) if $f(x) = e^{x^2}$,

$$f'(x) = (x^2)' \cdot e^{x^2} = 2x \cdot e^{x^2} = 2x \cdot f(x).$$

On the other hand, $\sin(x) \notin \mathbf{W}$ since if $f(x) = \sin(x)$,

$$f'(x) = \cos(x) \neq 2x \cdot \sin(x) = 2x \cdot f(x).$$

Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER: ______________________________

**2.**

[6 pts] (a) Use row operations to put the matrix below into Row Reduced Echelon Form (RREF).

$$\begin{bmatrix} 1 & 3 & 7 & 1 \\ 2 & -1 & -7 & 9 \\ 1 & 2 & 4 & 2 \end{bmatrix}$$

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER:

(— problem 2 continued —)

[4 pts] (b) Parametrize all the solutions to the system of linear equations below. Write your answer in vector form. [NOTE : Part (a) is relevant.]

$$\begin{array}{rcrcrcl} x & + & 3y & + & 7z & = & 1 \\ 2x & - & y & - & 7z & = & 9 \\ x & + & 2y & + & 4z & = & 2 \end{array}$$

[2 pts] (c) Find $\alpha_1$, $\alpha_2$, $\alpha_3 \in \mathbb{R}$ so that $(1, 9, 2) = \alpha_1(1, 2, 1) + \alpha_2(3, -1, 2) + \alpha_3(7, -7, 4)$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER:

**3.** Suppose that $L\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and we know that $L(5,2) = (3,2,-4)$ and that $L(2,1) = (1,2,-1)$.

[8 pts] (a) Find the standard matrix for $L$.

[4 pts] (b) Find $L(4,1)$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER:

**4.** The matrix $A$ and its RREF are shown below.

$$A = \begin{bmatrix} 2 & 1 & -4 & 3 & 9 \\ 1 & 5 & 7 & 2 & 1 \\ 3 & 0 & -9 & 0 & 6 \\ 1 & 1 & -1 & 0 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $L\colon \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ be the linear transformation whose standard matrix is $A$.

[4 pts] (a) Find a basis for $\mathrm{Im}(L)$.

[4 pts] (b) Find a basis for $\mathrm{Ker}(L)$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER: ___

**5.**

[5 pts] (a) Solve the system of linear equations below :

$$\begin{array}{rcrcrcl} 2x & + & y & + & 3z & = & 4 \\ x & & & + & z & = & 1 \\ & & 2y & + & 3z & = & 8 \end{array}$$

⟶ [**Note : Problem 5 continues on the back of this page**] ⟵

STUDENT NUMBER:

(— problem 5 continued —)

[5 pts] (b) Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$. Compute $A^{-1}$.

STUDENT NUMBER: ______________________________

(— problem 5 continued —)

[4 pts] (c) Let $\mathbf{w} = (4, 1, 8)$. Compute $A^{-1}\mathbf{w}$.

[4 pts] (d) Explain the connection between your answers in (a) and (c).

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER: ______________________________

**6.** Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

[6 pts] (a) Compute $\det(A)$.

[6 pts] (b) Compute $\det(B)$.

[2 pts] (c) Compute $\det(AB)$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER: ________________________

**7.** Let $L\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.

[2 pts] (a) State the rank-nullity theorem for $L$.

[2 pts] (b) State, without proof, what "$L$ is surjective" means in terms of the dimension of $\operatorname{Im}(L)$.

[2 pts] (c) State, without proof, what "$L$ is injective" means in terms of the dimension of $\operatorname{Ker}(L)$.

[4 pts] (d) Suppose we know that $L$ is surjective. State, <u>with proof</u>, which inequality must hold between $n$ and $m$. (I.e., which is bigger, and why.)

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER:

**8.** Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$.

[6 pts] (a) Verify that $\mathbf{v}_1 = (3, 2)$ and $\mathbf{v}_2 = (1, -1)$ are eigenvectors of $A$ and find their eigenvalues.

[2 pts] (b) Find $A^2\mathbf{v}_1$ and $A^2\mathbf{v}_2$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]

STUDENT NUMBER: ____________________

(— problem 8 continued —)

[4 pts] (c) Let $\mathbf{w} = (9, 1)$. Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2 pts] (d) For $n \geqslant 1$, find a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of $A$.

[2 pts] (e) Find $\lim\limits_{n\to\infty} A^n\mathbf{w}$.

STUDENT NUMBER:

[ Space for additional work. Indicate clearly which question you are continuing if you use this page. ]