STUDENT NUMBER:

Queen's University
APSC 174 – Final Exam
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HAND IN
answers recorded
on exam paper

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Solutions

Instructions:

The exam has **six** questions, worth a total of 100 marks.

Answer **all 6 questions**, writing clearly in the space provided, including the provided space for additional work. If you need more room, continue to answer on the **next blank page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order. Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed. Good luck!

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer the questions as written.

1	**2**	**3**	**4**	**5**	**6**	Total
/20	/15	/20	/15	/15	/15	/100

STUDENT NUMBER:

1. Consider a linear transformation $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ with standard matrix A and its row reduced row echelon form (RREF) given as follows:

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -9 & 4 \\ 0 & 2 & -6 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[3 pts] (a) Find a basis for $\text{Im}(L)$.

Solution. Since the leading ones appear in the first and second columns of the RREF of the standard matrix A, by our algorithm from class, the first and second columns of A form a basis for $\text{Im}(L)$. Thus $((2,1,0),(-1,3,2))$ is a basis for $\text{Im}(L)$.

Solution II. Using a theorem we learned in class, we know that the columns of A are a generating set for $\text{Im}(L)$. Since the first and second columns of A are linearly independent (neither is a scalar multiple of the other) and the third and fourth columns can each be written as a linear combination of the first two columns, we conclude that the third and fourth columns are redundant in the generating set. Hence the first and second columns form a basis for $\text{Im}(L)$. Thus $((2,1,0),(-1,3,2))$ is a basis for $\text{Im}(L)$.

[6 pts] (b) Find a basis for $\text{Ker}(L)$.

Solution. Adding an extra column of zeros to the RREF we get the matrix

$$\begin{array}{c} \begin{matrix} x & y & \;\;z & w & \;\; \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Using the algorithm to parameterize all solutions to the corresponding system of equations, we have here that the dependent variables are x and y, and that the free (independent) variables are z and w. Setting $z = t_1$ and $w = t_2$, we solve for x and y in terms of t_1 and t_2:

$$\begin{aligned} x &= -t_2 \\ y &= 3t_1 - t_2. \end{aligned}$$

Hence the general solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t_1 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

By our algorithm for finding a basis for $\text{Ker}(L)$, we obtain that

$$((0,3,1,0),(-1,-1,0,1))$$

is a basis for $\text{Ker}(L)$.

STUDENT NUMBER:

[4 pts] (c) Verify the Rank-Nullity theorem for L.

Solution. First note that

$\dim(\mathrm{Im}(L)) = 2$, since the basis for $\mathrm{Im}(L)$ in (a) has 2 vectors.

$\dim(\mathrm{Ker}(L)) = 2$, since the basis for $\mathrm{Ker}(L)$ in (b) has 2 vectors.

Now the rank-nullity theorem states that

$$\dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(\mathbb{R}^4) = 4.$$

This indeed holds since

$$\dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = 2 + 2 = 4.$$

[3 pts] (d) Is L injective (i.e., one-to-one)? Justify your answer.

Solution. We know from a theorem in class that the linear map is injective if and only if $\mathrm{Ker}(L) = \{\mathbf{0}\}$, where $\mathbf{0} = (0, 0, 0, 0)$ is the zero vector of the input (domain) vector space $\mathbb{R}^4$.

By (b), we have that $\mathrm{Ker}(L) \neq \{\mathbf{0}\}$ since for example $(0, 3, 1, 0) \in \mathrm{Ker}(L)$ or since

$$\dim(\mathrm{Ker}(L)) = 2 > 0 = \dim(\mathrm{Ker}(\{\mathbf{0}\})).$$

Hence the map L is *not injective.*

[4 pts] (e) Is L surjective (i.e., onto)? Justify your answer.

Solution. We have that

$$\mathrm{Im}(L) \neq \mathbb{R}^3$$

since $\dim(\mathrm{Im}(L)) = 2$ while $\dim(\mathbb{R}^3) = 3$. Hence the map L is *not surjective.*

STUDENT NUMBER:

2. Answer the following questions.

[8 pts] (a) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix}$.

Use the RREF algorithm to determine A^{-1} (show clearly your steps).

Solution. Applying the RREF algorithm to the augmented matrix $[A|I_3]$ where I_3 is the 3×3 identity matrix, we obtain that

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -12 & 3 & 5 \\ 0 & 1 & 0 & 8 & -2 & -3 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right] = [I_3|A^{-1}].$$

and thus the inverse of matrix A is

$$A^{-1} = \begin{bmatrix} -12 & 3 & 5 \\ 8 & -2 & -3 \\ -3 & 1 & 1 \end{bmatrix}.$$

A possible sequence of steps leading to the RREF above is as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 + R_2 \mapsto R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_3 \mapsto R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & -1 & -2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \mapsto R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_1 \mapsto R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_3 + R_2 \mapsto R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 0 & 8 & -2 & -3 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{5R_3 + R_1 \mapsto R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -12 & 3 & 5 \\ 0 & 1 & 0 & 8 & -2 & -3 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right].$$

STUDENT NUMBER:

[3 pts] (b) With the matrix A^{-1} obtained in (a), compute AA^{-1} to verify your answer.

Solution.
Method I: Let $\mathbf{w}_1$, $\mathbf{w}_1$, and $\mathbf{w}_3$ denote the columns of A^{-1}, then

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -12 & 3 & 5 \\ 8 & -2 & -3 \\ -3 & 1 & 1 \end{bmatrix} = (A\mathbf{w}_1 | A\mathbf{w}_2 | A\mathbf{w}_3),$$

where

$$\begin{aligned} A\mathbf{w}_1 &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -12 \\ 8 \\ -3 \end{bmatrix} = -12 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ A\mathbf{w}_2 &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ A\mathbf{w}_3 &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + (-3) \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, $AA^{-1} = I_3$.

Method II: Let $C = AA^{-1}$ and c_{ij} represents the entry of C in row i, column j. Then,

$$c_{ij} = (\text{Row } i \text{ of } A) \begin{bmatrix} \text{Column } j \\ \text{of} \\ A^{-1} \end{bmatrix} :$$

$$\begin{aligned} c_{11} = (1,2,1) \begin{bmatrix} -12 \\ 8 \\ -3 \end{bmatrix} = 1, \quad c_{12} = (1,2,1) \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0, \quad c_{13} = (1,2,1) \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = 0, \\ c_{21} = (1,3,4) \begin{bmatrix} -12 \\ 8 \\ -3 \end{bmatrix} = 0, \quad c_{22} = (1,3,4) \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 1, \quad c_{23} = (1,3,4) \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = 0, \\ c_{31} = (2,3,0) \begin{bmatrix} -12 \\ 8 \\ -3 \end{bmatrix} = 0, \quad c_{32} = (2,3,0) \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0, \quad c_{33} = (2,3,0) \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = 1. \end{aligned}$$

Therefore, $C = I_3$, that is, $AA^{-1} = I_3$.

STUDENT NUMBER:

[4 pts] (c) Solve for the unknowns x, y and z in terms of a, b and c in the following system of linear equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & a \\ x & + & 3y & + & 4z & = & b \\ 2x & + & 3y & & & = & c \end{array}$$

where a, b, and c are given scalars in $\mathbb{R}$.

Solution. The above system of linear equations can be written via the following matrix equation:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

where matrix A is given in part (a). Thus multiplying (from the left) both sides of the above equation by A^{-1} directly yields:

$$\underbrace{A^{-1}A}_{=I_3} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

and hence

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -12 & 3 & 5 \\ 8 & -2 & -3 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= a \begin{bmatrix} -12 \\ 8 \\ -3 \end{bmatrix} + b \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -12a + 3b + 5c \\ 8a - 2b - 3c \\ -3a + b + c \end{bmatrix}. \end{aligned}$$

Thus the unique solution of the above system of linear equations is given by

$$(x, y, z) = (-12a + 3b + 5c,\ 8a - 2b - 3c,\ -3a + b + c).$$

STUDENT NUMBER:

3. Answer the following questions.

[4 pts] (a) Given a vector space $\mathbf{V}$, state the definition of a vector subspace of $\mathbf{V}$.

Solution. A subset $\mathbf{W}$ of a vector space $\mathbf{V}$ is called a vector subspace (or subspace) of $\mathbf{V}$ if the following three tests hold under the operations of $\mathbf{V}$:

(i) The zero vector $\mathbf{0}$ of $\mathbf{V}$ is in $\mathbf{W}$: $\mathbf{0} \in \mathbf{W}$.

(ii) For any $\mathbf{u}$ and $\mathbf{v} \in \mathbf{W}$, $\mathbf{u} + \mathbf{v} \in \mathbf{W}$.

(iii) For any $\alpha \in \mathbb{R}$ and $\mathbf{v} \in \mathbf{W}$, $\alpha\mathbf{v} \in \mathbf{W}$.

[4 pts] (b) State the definition of a linear transformation between two vector spaces $\mathbf{V}$ and $\mathbf{W}$.

Solution. A function $L : \mathbf{V} \to \mathbf{W}$ between two vector spaces $\mathbf{V}$ and $\mathbf{W}$ is called a linear transformation if the following properties hold:

(i) For any $\mathbf{u}$ and $\mathbf{v} \in \mathbf{V}$, $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$.

(ii) For any $\alpha \in \mathbb{R}$ and $\mathbf{v} \in \mathbf{V}$, $L(\alpha\mathbf{v}) = \alpha L(\mathbf{v})$.

[12 pts] (c) For each of the following statements indicate (without proof) whether it is *True* (T) or *False* (F).

	T or F
(i) The union of two vector subspaces of $\mathbb{R}^3$ is a vector subspace of $\mathbb{R}^3$.	F
(ii) In a vector space, subsets of linear independent sets are also linearly independent.	T
(iii) The set of all polynomial functions of the form $p(t) = at^2 + b$, $a, b \in \mathbb{R}$, is a vector subspace of the vector space $P_3(\mathbb{R})$ (of cubic polynomial functions from $\mathbb{R}$ to $\mathbb{R}$).	T
(iv) Every system of linear equations has a solution.	F
(v) The function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(x) = (2x, x+1)$ is a linear transformation.	F
(vi) If A and B are 2×2 matrices, then $\det(A + 2B) = \det(A) + 2\det(B)$.	F

STUDENT NUMBER:

4. Consider the matrix

$$A = \begin{bmatrix} 1 & -4 & 6 \\ 0 & -3 & 6 \\ 0 & -4 & 7 \end{bmatrix}.$$

[5 pts] (a) Find the characteristic polynomial of A, and verify that 1 and 3 are roots of the polynomial.

Solution. (a) The characteristic polynomial of A is given by $\det(A - \lambda I_3)$, also written as $|A - \lambda I_3|$:

$$\begin{aligned} \left| \begin{bmatrix} 1-\lambda & -4 & 6 \\ 0 & -3-\lambda & 6 \\ 0 & -4 & 7-\lambda \end{bmatrix} \right| &= (1-\lambda)[(-3-\lambda)(7-\lambda) - (-4)(6)] \\ &= (1-\lambda)(-21 - 4\lambda + \lambda^2 + 24) \\ &= (1-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= (1-\lambda)(\lambda - 1)(\lambda - 3) \\ \text{or optionally} &= -(\lambda-1)^2(\lambda - 3) \end{aligned}$$

The roots of the characteristic polynomial are $\lambda = 1$ (multiplicity 2), and $\lambda = 3$.
Note: students could also sub in the test values $\lambda = 1$ or $\lambda = 3$ into an earlier line, and note that the value of the polynomial is zero there; this would also show that $\lambda = 1$ and $\lambda = 3$ are roots of the characteristic polynomial.

[5 pts] (b) Find a basis for the eigenspace of vectors corresponding to eigenvalue 1 of A.

Solution. To find a basis for the eigenspace for $\lambda = 1$, we look for possible eigenvectors with eigenvalue $\lambda = 1$. These non-trival vectors $\mathbf{v}$ satisfy the equation $(A - 1 \cdot I_3)\mathbf{v} = \mathbf{0}$.

$$\begin{bmatrix} 1-1 & -4 & 6 \\ 0 & -3-1 & 6 \\ 0 & -4 & 7-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & 6 \\ 0 & -4 & 6 \\ 0 & -4 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting in RREF for ease of analysis,

$$\begin{matrix} R_1/(-4) \\ R_1 - R_2 \\ R_1 - R_3 \end{matrix} \begin{bmatrix} 0 & 1 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables in this system of equations are v_1 and v_3, and $v_2 = (\frac{3}{2})v_3$. In vector form, with $v_1 = s$ and $v_3 = t$, we get

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ \frac{3}{2}t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \end{bmatrix} t$$

STUDENT NUMBER:

Thus one basis for the eigenspace for $\lambda = 1$ is the pair of vectors

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{3}{2} \\ 1 \end{bmatrix} \right).$$

[5 pts] (c) Find a basis for the eigenspace of vectors corresponding to eigenvalue 3 of A.

Solution. To find a basis for the eigenspace for $\lambda = 3$, we look for possible eigenvectors with eigenvalue $\lambda = 3$. These non-trival vectors $\mathbf{v}$ satisfy the equation $(A - 3 \cdot I_3)\mathbf{v} = \mathbf{0}$.

$$\begin{bmatrix} 1-3 & -4 & 6 \\ 0 & -3-3 & 6 \\ 0 & -4 & 7-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -4 & 6 \\ 0 & -6 & 6 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting in RREF for ease of analysis,

$$\begin{array}{c} R_1/(-2) \\ R_1 - R_3 \\ R_2/6 - R_3/4 \end{array} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{array}{c} R_1 - 2R_2 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The only free variable is v_3, and the first two lines in the RREF matrix can be rearranged to find $v_1 = v_3$ and $v_2 = v_3$.
In vector form, with $v_3 = t$, we get

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$

Thus a basis for the eigenspace for $\lambda = 3$ is the vector

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

STUDENT NUMBER:

5. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 0 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix} .$$

[5 pts] (a) Calculate $\det(A)$ and $\det(B)$.

Solution 1. Using Sarrus' rule we calculate

$$\begin{aligned} \det(A) &= \det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 2 \\ 2 & 3 & -1 & 2 & 3 \\ 1 & 0 & 3 & 1 & 0 \end{array} \\ &= 1\times 3\times 3 + 2\times(-1)\times 1 + (-1)\times 2\times 0 - (-1)\times 3\times 1 - 1\times(-1)\times 0 - 2\times 2\times 3 \\ &= 9 - 2 + 0 + 3 - 0 - 12 = -2 \end{aligned}$$

and similarly

$$\begin{aligned} \det(B) &= \det \begin{bmatrix} 2 & -1 & 0 \\ 5 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \begin{array}{ccc|cc} 2 & -1 & 0 & 2 & -1 \\ 5 & 3 & 2 & 5 & 3 \\ 1 & 1 & 3 & 1 & 1 \end{array} \\ &= 2\times 3\times 3 + (-1)\times 2\times 1 + 0\times 5\times 1 - 0\times 3\times 1 - 2\times 2\times 1 - (-1)\times 5\times 3 \\ &= 18 - 2 + 0 - 0 - 4 + 15 = 27 . \end{aligned}$$

Solution 2. Expanding along the first row, we obtain

$$\begin{aligned} \det(A) &= \det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 0 & 3 \end{bmatrix} = 1 \ \det \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix} - 2 \ \det \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + (-1) \ \det \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \\ &= 1(3\times 3 - (-1)\times 0) - 2(2\times 3 - (-1)\times 1) - 1(2\times 0 - 3\times 1) = 9 - 2\times 7 - (-3) = -2 \end{aligned}$$

and similarly

$$\begin{aligned} \det(B) &= \det \begin{bmatrix} 2 & -1 & 0 \\ 5 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix} = 2 \ \det \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} - (-1) \ \det \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} + 0 \ \det \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} \\ &= 2(3\times 3 - 2\times 1) + (5\times 3 - 2\times 1) + 0 = 2\times 7 + 13 = 27 . \end{aligned}$$

Possible variations include expanding $\det(A)$ along the third row or second column to use the zero.

STUDENT NUMBER:

[5 pts] (b) Find $\det(A^2)$ and $\det(AB^{-1})$.

Solution. Using properties of determinants we calculate

$$\det(A^2) = \det(A\,A) = \det(A)\ \det(A) = (-2) \times (-2) = 4$$

and

$$\det(AB^{-1}) = \det(A)\ \det(B^{-1}) = \det(A)\,\frac{1}{\det(B)} = -2 \times \frac{1}{27} = -\frac{2}{27}\ .$$

[5 pts] (c) Calculate the determinant of the following matrix:

$$C = \begin{bmatrix} 5 & 0 & 1 & 8 \\ 2 & 0 & 0 & 5 \\ 2 & 2 & -3 & 6 \\ 3 & 0 & 0 & 7 \end{bmatrix}\ .$$

Solution. Expanding along the second column and, for the resulting matrix expanding along the second column, we calculate

$$\det(C) = \det\begin{bmatrix} 5 & 0 & 1 & 8 \\ 2 & 0 & 0 & 5 \\ 2 & 2 & -3 & 6 \\ 3 & 0 & 0 & 7 \end{bmatrix} = -2\ \det\begin{bmatrix} 5 & 1 & 8 \\ 2 & 0 & 5 \\ 3 & 0 & 7 \end{bmatrix} = -2 \times (-1) \det\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} = 2(2 \times 7 - 3 \times 5) = -2\ .$$

STUDENT NUMBER:

6. Answer the following questions.

[8 pts] (a) Given a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, show that L is injective (i.e., one-to-one) if and only if $\text{Ker}(L) = \{\mathbf{0}_{\mathbb{R}^n}\}$, where $\mathbf{0}_{\mathbb{R}^n}$ is the zero vector of $\mathbb{R}^n$.

Solution. We need to prove two directions.
$\Rightarrow$: Assume first that L is injective. We will show that $\text{Ker}(L) = \{\mathbf{0}_{\mathbb{R}^n}\}$.

(i) Since $\text{Ker}(L)$ is a subspace of $\mathbb{R}^n$, it contains the zero vector and hence $\{\mathbf{0}_{\mathbb{R}^n}\} \subset \text{Ker}(L)$.

(ii) If $\mathbf{v} \in \text{Ker}(L)$, then
$$L(\mathbf{v}) = \mathbf{0}_{\mathbb{R}^m} = L(\mathbf{0}_{\mathbb{R}^n}) \ .$$
Since $L(\mathbf{v}) = L(\mathbf{0}_{\mathbb{R}^n})$ and L is injective, we conclude that $\mathbf{v} = \mathbf{0}_{\mathbb{R}^n}$, proving that $\text{Ker}(L) \subset \{\mathbf{0}_{\mathbb{R}^n}\}$.

The inclusions $\{\mathbf{0}_{\mathbb{R}^n}\} \subset \text{Ker}(L)$ and $\text{Ker}(L) \subset \{\mathbf{0}_{\mathbb{R}^n}\}$ imply that $\text{Ker}(L) = \{\mathbf{0}_{\mathbb{R}^n}\}$.

$\Leftarrow$: Assume now that $\text{Ker}(L) = \{\mathbf{0}_{\mathbb{R}^n}\}$. We will show that L is injective.
Assume that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are such that $L(\mathbf{u}) = L(\mathbf{v})$. By linearity of L we have
$$L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v}) = \mathbf{0}_{\mathbb{R}^m} \ ,$$
which implies that $\mathbf{u} - \mathbf{v} \in \text{Ker}(L)$. Since $\text{Ker}(L) = \{\mathbf{0}_{\mathbb{R}^n}\}$, we conclude that $\mathbf{u} - \mathbf{v} = \mathbf{0}_{\mathbb{R}^n}$, which leads to $\mathbf{u} = \mathbf{v}$. This completes the proof that L is injective.

[7 pts] (b) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two eigenvectors of an $n \times n$ matrix A with corresponding eigenvalues λ_1 and λ_2, respectively. Show that if $\lambda_1 \neq \lambda_2$, then the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Solution 1. To prove that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, assume that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^n}$. Multiplying by A and using the property of eigenvectors, we obtain
$$\mathbf{0}_{\mathbb{R}^n} = A\,\mathbf{0}_{\mathbb{R}^n} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 \ .$$
Multiplying $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^n}$ by λ_2 we obtain $c_1\lambda_2\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^n}$. Now subtracting two expressions that equal to zero, we get
$$\mathbf{0}_{\mathbb{R}^n} = \mathbf{0}_{\mathbb{R}^n} - \mathbf{0}_{\mathbb{R}^n} = (c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2) - (c_1\lambda_2\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2) = c_1(\lambda_1 - \lambda_2)\mathbf{v}_1 \ .$$
The vector $\mathbf{v}_1$ is an eigenvector and thus, by definition, it is not the zero vector. Hence $c_1(\lambda_1 - \lambda_2) = 0$. But $\lambda_1 - \lambda_2 \neq 0$ because $\lambda_1 \neq \lambda_2$. This implies that $c_1 = 0$. Substituting $c_1 = 0$ into $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^n}$ we obtain $c_2\mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^n}$. Using that $\mathbf{v}_2$ is not the zero vector (being an eigenvector), we conclude that $c_2 = 0$.

To summarize: the assumption that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}_{\mathbb{R}^n}$ implies that $c_1 = c_2 = 0$, proving that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

STUDENT NUMBER:

Solution 2. Assume, to the contrary, that $\lambda_1 \neq \lambda_2$ and the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent. Then one of the vectors $\mathbf{v}_1$ is a scalar multiple of the other one. Without loss of generality we may assume that $\mathbf{v}_2 = c\mathbf{v}_1$. Then we calculate $A\mathbf{v}_2$ in two different ways:

$$\begin{aligned} A\mathbf{v}_2 \quad &= \lambda_2\mathbf{v}_2 = \lambda_2(c\mathbf{v}_1) = (c\lambda_2)\mathbf{v}_1 \\ A\mathbf{v}_2 \quad &= A(c\mathbf{v}_1) = c\,A\mathbf{v}_1 = c(\lambda_1\mathbf{v}_1) = (c\lambda_1)\mathbf{v}_1 \ . \end{aligned}$$

Comparing the righthand sides we get $(c\lambda_2)\mathbf{v}_1 = (c\lambda_2)\mathbf{v}_1$. Subtracting the two we obtain $c(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}_{\mathbb{R}^n}$. Arguing as in Solution 1, we conclude that $c(\lambda_1 - \lambda_2) = 0$ and hence $c = 0$. But this implies that $\mathbf{v}_2 = c\mathbf{v}_1 = \mathbf{0}_{\mathbb{R}^n}$ which contradicts the assumption that $\mathbf{v}_2$ is an eigenvector and hence a nonzero vector. This contradiction completes the proof that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.
