STUDENT NUMBER:

Queen's University
APSC 174 – Final Exam
April 2022

HAND IN
answers recorded
on exam paper

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Instructions:

The exam has **six** questions, worth a total of 100 marks.

Answer **all 6 questions**, writing clearly in the space provided, including the provided space for additional work. If you need more room, continue to answer on the **next blank page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order. Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed. Good luck!

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer the questions as written.

1	**2**	**3**	**4**	**5**	**6**	Total
/20	/15	/20	/15	/15	/15	/100

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1. Consider a linear transformation $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ with standard matrix A and its row reduced row echelon form (RREF) given as follows:

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 3 & -9 & 4 \\ 0 & 2 & -6 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[3 pts] (a) Find a basis for $\operatorname{Im}(L)$.

[6 pts] (b) Find a basis for $\operatorname{Ker}(L)$.

[4 pts] (c) Verify the Rank-Nullity theorem for L.

[3 pts] (d) Is L injective (i.e., one-to-one)? Justify your answer.

[4 pts] (e) Is L surjective (i.e., onto)? Justify your answer.

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2. Answer the following questions.

[8 pts] (a) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix}$.

Use the RREF algorithm to determine A^{-1} (show clearly your steps).

[3 pts] (b) With the matrix A^{-1} obtained in (a), compute AA^{-1} to verify your answer.

[4 pts] (c) Solve for the unknowns x, y and z in terms of a, b and c in the following system of linear equations

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & a \\ x & + & 3y & + & 4z & = & b \\ 2x & + & 3y & & & = & c \end{array}$$

where a, b, and c are given scalars in $\mathbb{R}$.

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3. Answer the following questions.

[4 pts] (a) Given a vector space $\mathbf{V}$, state the definition of a vector subspace of $\mathbf{V}$.

[4 pts] (b) State the definition of a linear transformation between two vector spaces $\mathbf{V}$ and $\mathbf{W}$.

[12 pts] (c) For each of the following statements indicate (without proof) whether it is *True* (T) or *False* (F).

	T or F
(i) The union of two vector subspaces of $\mathbb{R}^3$ is a vector subspace of $\mathbb{R}^3$.	______
(ii) In a vector space, subsets of linear independent sets are also linearly independent.	______
(iii) The set of all polynomial functions of the form $p(t) = at^2 + b$, $a, b \in \mathbb{R}$, is a vector subspace of the vector space $P_3(\mathbb{R})$ (of cubic polynomial functions from $\mathbb{R}$ to $\mathbb{R}$).	______
(iv) Every system of linear equations has a solution.	______
(v) The function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(x) = (2x, x+1)$ is a linear transformation.	______
(vi) If A and B are 2×2 matrices, then $\det(A + 2B) = \det(A) + 2\det(B)$.	______

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4. Consider the matrix

$$A = \begin{bmatrix} 1 & -4 & 6 \\ 0 & -3 & 6 \\ 0 & -4 & 7 \end{bmatrix}.$$

[5 pts] (a) Find the characteristic polynomial of A, and verify that 1 and 3 are roots of the polynomial.

[5 pts] (b) Find a basis for the eigenspace of vectors corresponding to eigenvalue 1 of A.

[5 pts] (c) Find a basis for the eigenspace of vectors corresponding to eigenvalue 3 of A.

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5. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 0 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

[5 pts] (a) Calculate $\det(A)$ and $\det(B)$.

[5 pts] (b) Find $\det(A^2)$ and $\det(AB^{-1})$.

[5 pts] (c) Calculate the determinant of the following matrix:

$$C = \begin{bmatrix} 5 & 0 & 1 & 8 \\ 2 & 0 & 0 & 5 \\ 2 & 2 & -3 & 6 \\ 3 & 0 & 0 & 7 \end{bmatrix}.$$

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6. Answer the following questions.

[8 pts] (a) Given a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, show that L is injective (i.e., one-to-one) if and only if $\mathrm{Ker}(L) = \{\mathbf{0}_{\mathbb{R}^n}\}$, where $\mathbf{0}_{\mathbb{R}^n}$ is the zero vector of $\mathbb{R}^n$.

[7 pts] (b) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two eigenvectors of an $n \times n$ matrix A with corresponding eigenvalues λ_1 and λ_2, respectively. Show that if $\lambda_1 \neq \lambda_2$, then the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

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