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APSC 174 — Midterm 1

Monday February 11, 2019

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First name (please write as legibly as possible within the boxes)

Last name

Student ID number

Instructions: The exam has six questions, worth a total of 100 marks.

Answer **all questions**, writing clearly in the space provided. If you need more room, continue to answer on the back of the **previous page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order.

No textbook, lecture notes, calculator, computer, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	**6**	Total
/15	/20	/15	/15	/20	/15	/100

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1. Consider the set
$$\mathbf{V} = \{(x, y) \ : \ x, y \in \mathbb{R},\ y > 0\}$$
with the addition and scalar multiplication rules given by

Addition: For any (x_1, y_1), $(x_2, y_2) \in \mathbf{V}$,
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 - 2,\ y_1 y_2).$$

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $(x, y) \in \mathbf{V}$,
$$\alpha \cdot (x, y) = (\alpha x - 2\alpha + 2,\ y^\alpha).$$

It can be proved (and you do not have to do this) that $\mathbf{V}$ with these operations is a vector space.

[5 pts] (a) Compute (using the operations on $\mathbf{V}$) the linear combination $2 \cdot (3, 5) + 5 \cdot (4, 1)$.

Solution. Using the operations of addition and multiplications in $\mathbf{V}$ defined above, we have

$$\begin{aligned} 2 \cdot (3, 5) + 5 \cdot (4, 1) &= (2 \cdot 3 - 2 \cdot 2 + 2, 5^2) + (5 \cdot 4 - 2 \cdot 5 + 2, 1^5) \\ &= (4, 25) + (12, 1) \\ &= (4 + 12 - 2, 25 \cdot 1) \\ &= \underline{(14, 25)}. \end{aligned}$$

[5 pts] (b) Determine the zero vector $\mathbf{0}$ of $\mathbf{V}$.

Solution.
Argument 1: We have learned in class that if $\mathbf{v}$ is any vector in a vector space $\mathbf{V}$, then $0 \cdot \mathbf{v} = \mathbf{0}$. If we choose (say) $(1, 1) \in \mathbf{V}$, then we obtain
$$0 \cdot (1, 1) = (0 \cdot 1 - 2 \cdot 0 + 2, 1^0) = (2, 1)$$
so that $\underline{\mathbf{0} = (2, 1)}$.

Argument 2: We can also determine $\mathbf{0}$ by the requirement (from the Axioms) that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for any vector $\mathbf{v} \in \mathbf{V}$. Letting $(x, y) \in \mathbf{V}$ be arbitrary and denoting components of $\mathbf{0}$ by x_0 and y_0 (so that $\mathbf{0} = (x_0, y_0)$), we obtain the equation
$$(x, y) + (x_0, y_0) = (x, y)$$
or, using the addition operation in $\mathbf{V}$,
$$(x + x_0 - 2, y y_0) = (x, y).$$
Equating the corresponding entries of the two vectors, we get $x + x_0 - 2 = x$ or $x_0 = 2$ and $y y_0 = y$ or $y_0 = 1$. Thus $\mathbf{0} = (2, 1)$.

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[5 pts] (c) For $(x, y) \in \mathbf{V}$, determine its additive inverse; that is, find a vector (x', y') such that $(x, y) + (x', y') = \mathbf{0}$.

Solution.

Argument 1: Here we can us the fact, learned in class, that the additive inverse of any $(x, y) \in \mathbf{V}$ is $(-1) \cdot (x, y)$. Accordingly,

$$(x', y') = (-1) \cdot (x, y) = ((-1) \cdot x - 2 \cdot (-1) + 2, y^{-1}) = (-x + 4, 1/y)$$

so that the additive inverse of (x, y) is $\underline{(-x + 4, 1/y)}$.

Argument 2:

Alternatively, we can find the additive inverse (x', y') of $(x, y) \in \mathbf{V}$ from the requirement that $(x, y) + (x', y') = \mathbf{0}$. We have

$$(x, y) + (x', y') = (x + x' - 2, yy')$$

and since $\mathbf{0} = (1, 2)$ we obtain the vector equation

$$(x + x' - 2, yy') = (2, 1)$$

i.e., $x + x' - 2 = 2$ and $yy' = 1$. Solving for x' and y' gives $x' = -x + 4$ and $y' = 1/y$, i.e., $(x', y') = \underline{(-x + 4, 1/y)}$.

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2. Let
$$\mathbf{W} = \Big\{(x, y, z) \in \mathbb{R}^3 \ : \ 3x - 2y + z = 0\Big\} \subset \mathbb{R}^3.$$
Here the operations on $\mathbb{R}^3$ are the usual ones of addition and scalar multiplication.

[10 pts] (a) Show, with proof, that $\mathbf{W}$ is a subspace of $\mathbb{R}^3$.

Solution. We check the conditions to be a subspace.

(i) **Q:** Is $\mathbf{0} \in \mathbf{W}$?

A: Yes. In $\mathbb{R}^3$, $\mathbf{0} = (0, 0, 0)$ and since $3 \cdot 0 - 2 \cdot 0 + 0 = 0$, we conclude that $\mathbf{0} \in \mathbf{W}$.

(ii) **Q:** If $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{W}$, is $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}$?

A: Yes. Suppose that $\mathbf{w}_1 = (x_1, y_1, z_1)$. Since $\mathbf{w}_1 \in \mathbf{W}$ we know that $3x_1 - 2y_1 + z_1 = 0$. Similarly, let $\mathbf{w}_2 = (x_2, y_2, z_2)$. Since $\mathbf{w}_2 \in \mathbf{W}$ we know that $3x_2 - 2y_2 + z_2 = 0$.

Then $\mathbf{w}_1 + \mathbf{w}_2 = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$, and this vector is in $\mathbf{W}$ since
$$\begin{aligned} 3(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2) &= 3x_1 + 3x_2 - 2y_1 - 2y_2 + z_1 + z_2 \\ &= (3x_1 - 2y_1 + z_1) + (3x_2 - 2y_2 + z_2) \\ &= 0 + 0 = 0. \end{aligned}$$

(iii) **Q:** If $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbf{W}$, is $\alpha \cdot \mathbf{w} \in \mathbf{W}$?

A: Yes. Let $\mathbf{w} = (x, y, z)$. Since $\mathbf{w} \in \mathbf{W}$ we know that $3x - 2y + z = 0$. Then $\alpha \cdot \mathbf{w} = (\alpha x,\ \alpha y,\ \alpha z)$, and $\alpha \cdot \mathbf{w} \in \mathbf{W}$ since
$$3(\alpha x) - 2(\alpha y) + (\alpha z) = \alpha(3x - 2y + z) = \alpha \cdot 0 = 0.$$

Since $\mathbf{W}$ passes all three tests, $\mathbf{W}$ is a subspace of $\mathbb{R}^3$.

[10 pts] (b) Let $\mathbf{v}_1 = (1, 0, -3)$ and $\mathbf{v}_2 = (0, 1, 2)$. Show that $S_{(\mathbf{v}_1, \mathbf{v}_2)} \subset \mathbf{W}$.

Solution.
Argument 1: Since $3 \cdot 1 - 2 \cdot 0 + 1 \cdot (-3) = 0$ and $3 \cdot 0 - 2 \cdot 1 + 1 \cdot 2 = 0$ we have that $\mathbf{v}_1$ and $\mathbf{v}_2 \in \mathbf{W}$. We know that $\mathbf{W}$ is a subspace of $\mathbb{R}^3$ by (a). Using condition (iii) to be a subspace we conclude that for any α_1, $\alpha_2 \in \mathbb{R}$, $\alpha_1 \cdot \mathbf{v}_1$ and $\alpha_2 \cdot \mathbf{v}_2$ are both in $\mathbf{W}$. Using condition (ii) to be a subspace we then conclude that $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 \in \mathbf{W}$.

Therefore, all vectors of the form $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2$ are in $\mathbf{W}$. The set of vectors of this form is exactly $S_{(\mathbf{v}_1, \mathbf{v}_2)}$, and so $S_{(\mathbf{v}_1, \mathbf{v}_2)} \subset \mathbf{W}$.

Argument 2: The vectors in $S_{(\mathbf{v}_1, \mathbf{v}_2)}$ are all the vectors of the form $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2$ with α_1, $\alpha_2 \in \mathbb{R}$. Expanding, we have
$$\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 = \alpha_1(1, 0, -3) + \alpha_2(0, 1, 2) = (\alpha_1,\ \alpha_2, -3\alpha_1 + 2\alpha_2).$$
Applying the test to be in $\mathbf{W}$ to this vector we get
$$3\alpha_1 - 2\alpha_2 + (-3\alpha_1 + 2\alpha_2) = (3\alpha_1 - 3\alpha_1) + (-2\alpha_2 + 2\alpha_2) = 0.$$
Therefore no matter what α_1 and α_2 are, $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 \in \mathbf{W}$, and so $S_{(\mathbf{v}_1, \mathbf{v}_2)} \subset \mathbf{W}$.

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3. Recall that $C^\infty(\mathbb{R})$ is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ which have a first, second, third, $\ldots$, etc, derivative. The operations on $C^\infty(\mathbb{R})$ are addition and scalar multiplication of functions as used in class. Let

$$\mathbf{W} = \left\{ f \in C^\infty(\mathbb{R}) \ : \ \int_{-\pi}^{\pi} f(x)\,dx = 0 \right\} \subset C^\infty(\mathbb{R}).$$

For instance, $x \in \mathbf{W}$ since

$$\int_{-\pi}^{\pi} x\,dx = \frac{x^2}{2}\bigg|_{x=-\pi}^{x=\pi} = \frac{1}{2}\left((\pi)^2 - (-\pi)^2\right) = 0,$$

while $x^2 \notin \mathbf{W}$ since $\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^3 \neq 0$.

[5 pts] (a) Is $\sin(x) \in \mathbf{W}$?

Solution. Yes. Since $\sin(x)$ has derivatives of arbitrary order, we have $\sin(x) \in C^\infty(\mathbb{R})$. Since

$$\int_{-\pi}^{\pi} \sin(x)\,dx = -\cos(x)\bigg|_{x=-\pi}^{x=\pi} = \big(-\cos(\pi) + \cos(-\pi)\big) = \big(1-1\big) = 0,$$

we obtain that $\underline{\sin(x) \in \mathbf{W}}$.

[10 pts] (b) Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

Solution. We check the three conditions for $\mathbf{W}$ to be a subspace:

(i) **Q:** Is $\mathbf{0} \in \mathbf{W}$?

A: Yes. Since the zero vector $\mathbf{0}$ in $\mathbf{W}$ is the all zero function $f(x) = 0$ for all $x \in \mathbb{R}$ and since

$$\int_{-\pi}^{\pi} 0\,dx = 0,$$

we indeed have $\underline{\mathbf{0} \in \mathbf{W}}$.

(ii) **Q:** If $f_1, f_2 \in \mathbf{W}$, is $f_1 + f_2 \in \mathbf{W}$?

A: Yes. Since $f_1, f_2 \in \mathbf{W}$, we know that $\int_{-\pi}^{\pi} f_1(x)\,dx = 0$ and $\int_{-\pi}^{\pi} f_2(x)\,dx = 0$, and so

$$\int_{-\pi}^{\pi} (f_1+f_2)(x)\,dx = \int_{-\pi}^{\pi} \big(f_1(x) + f_2(x)\big)\,dx = \int_{-\pi}^{\pi} f_1(x)\,dx + \int_{-\pi}^{\pi} f_2(x)\,dx = 0.$$

Thus $\underline{f_1 + f_2 \in \mathbf{W}}$.

(iii) **Q:** If $\alpha \in \mathbb{R}$ and $f \in \mathbf{W}$, is $\alpha f \in \mathbf{W}$?

A: Yes. Since $f \in \mathbf{W}$ implies $\int_{-\pi}^{\pi} f(x)\,dx = 0$, we have

$$\int_{-\pi}^{\pi} (\alpha f)(x)\,dx = \int_{-\pi}^{\pi} \alpha f(x)\,dx = \alpha \int_{-\pi}^{\pi} f(x)\,dx = 0,$$

which means that $\underline{\alpha f \in \mathbf{W}.}$

Since $\mathbf{W}$ satisfies all three conditions, $\underline{\mathbf{W}\text{ is a subspace}}$.

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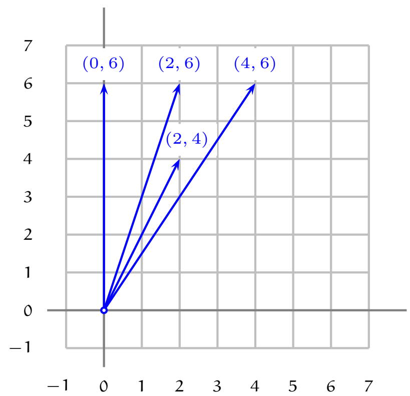
4.

[5 pts] (a) In the grid below, draw and label the vector $(2,4)$ and the vectors $(0,6)$, $(2,6)$, and $(4,6)$.

Solution.



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[10 pts] (b) For which values of $t \in \mathbb{R}$ do the vectors $(2,4)$ and $(t,6)$ span $\mathbb{R}^2$?

Solution.

Argument 1 (Geometric): If $t = 3$ then the vectors $(2,4)$ and $(t,6) = (3,6)$ are both on the line $y = 2x$, and so the span of the vectors is that line and not all of $\mathbb{R}^2$.

On the other hand if $t \neq 3$ then the vectors $(2,4)$ and $(t,6)$ are not on the same line, and so linearly independent (two vectors are linearly dependent if and only if one is a scalar multiple of the other, i.e., if and only if they lie on the same line through the origin). These two vectors linearly independent vectors therefore span all of $\mathbb{R}^2$.

Therefore the vectors $(2,4)$ and $(t,6)$ span $\mathbb{R}^2$ if and only if $t \neq 3$.

Argument 2 (Algebraic): The vectors $(2,4)$ and $(t,6)$ span $\mathbb{R}^2$ if any vector $(x,y) \in \mathbb{R}^2$ can be written as a linear combination of $(2,4)$ and $(t,6)$. That is, we have to check if for an arbitrary $(x,y) \in \mathbb{R}^2$ there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} t \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This vector equation is equivalent to the system of linear equations

$$\begin{aligned} 2\alpha + t\beta &= x \\ 4\alpha + 6\beta &= y. \end{aligned}$$

To solve this system for α and β, multiply the first equation by 2 and subtract it from the second equation to obtain

$$6\beta - 2t\beta = y - 2x. \tag{$*$}$$

If $6 - 2t \neq 0$, i.e., if $t \neq 3$, we can solve for β as

$$\boxed{\beta = \frac{y - 2x}{6 - 2t}}$$

On the other hand, substituting the formula for β into the second equation gives us

$$4\alpha + 6\left(\frac{y - 2x}{6 - 2t}\right) = y$$

and solving for α gives us

$$\alpha = \frac{1}{4}\left(y - 6\left(\frac{y - 2x}{6 - 2t}\right)\right) = \frac{1}{4}\left(\frac{6y - 2ty}{6 - 2t} - \frac{6y - 12x}{6 - 2t}\right) = \frac{1}{4} \cdot \frac{12x - 2ty}{6 - 2t} = \frac{6x - ty}{12 - 4t}$$

Thus if $t \neq 3$, then for any $(x,y) \in \mathbb{R}^2$ the system has a (unique) solution for α and β. This proves that for all values of $t \in \mathbb{R}$ not equal to 3, the vectors $(2,4)$ and $(t,6)$ span $\mathbb{R}^2$.

On the other hand, if $t = 3$ then the left hand side of equation $(*)$ is zero, giving $y = 2x$. Thus if $t = 3$, the system cannot be solved for an arbitrary (x,y) which means that in this case, the vectors $(2,4)$ and $(t,6)$, i.e., $(2,4)$ and $(3,6)$ do not span $\mathbb{R}^2$.

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5.

[5 pts] (a) State what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ to be *linearly independent*.

Solution. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if the only solution of the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_p \mathbf{v}_p = \mathbf{0}$$

is $\alpha_1 = 0,\ \alpha_2 = 0,\ \ldots \alpha = 0$. In other words, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if the only way to write the zero vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is with all zero coefficients.

[5 pts] (b) State what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ to be *linearly dependent*.

Solution. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent if the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_p \mathbf{v}_p = \mathbf{0}$$

has solutions other than $\alpha_1 = 0,\ \alpha_2 = 0,\ \ldots \alpha = 0$. In other words, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent if the zero vector can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with coefficients that are not all zero.

[10 pts] (c) Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, are elements of a vector space $\mathbf{V}$, and we know that the set $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is linearly *independent*, and also that the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is linearly *dependent*. Show that $\mathbf{v}_3 \in S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

Solution. Since $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is linearly *dependent* we know from part (b) that there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0}. \qquad (**)$$

If $\alpha_3 = 0$, then we must have either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Thus if $\alpha_3 = 0$, we obtain

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \mathbf{0}.$$

with α_1 and α_2 not both zero. But this contradicts the assumption that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Thus in equation $(**)$ we must have $\alpha_3 \neq 0$. Then multiplying both sides by $\frac{1}{\alpha_3}$ and rearranging the equation gives

$$\mathbf{v}_3 = -\frac{\alpha_1}{\alpha_3}\mathbf{v}_1 - \frac{\alpha_2}{\alpha_3}\mathbf{v}_2.$$

Thus $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, which means that $\mathbf{v}_3 \in S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

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6. Let $\mathbf{v}_1 = (4,3,1)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (0,1,-1)$, and $\mathbf{v}_4 = (2,0,3)$ in $\mathbb{R}^3$.

[5 pts] (a) Is $\mathbf{v}_1$ a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$?

Solution. Yes. We have to see if there exist $\alpha, \beta \in \mathbb{R}$ which give $\alpha\mathbf{v}_2 + \beta\mathbf{v}_3 = \mathbf{v}_1$. In column vector form this equation is

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

This vector equation is equivalent to the system

$$\begin{aligned} \alpha \quad &= 4 \\ \beta \quad &= 3 \\ \alpha - \beta &= 1 \end{aligned}$$

From the first two equations we get $\underline{\alpha = 4}$ and $\underline{\beta = 3}$, and since these values also solve the third equation, $\underline{\mathbf{v}_1 \text{ is a linear combinations of } \mathbf{v}_2 \text{ and } \mathbf{v}_3}$: $\mathbf{v}_1 = 4\mathbf{v}_2 + 3\mathbf{v}_3$.

[5 pts] (b) Determine, with proof, whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

Solution.

Argument 1: We know from class that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent if and only if one of these vectors is a linear combination of the other two. Since $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ by part (a), these three vectors are linearly dependent.

Argument 2: For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent the only solution to

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$$

should be $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$. But we know from part (a) that $\mathbf{v}_1 = 4\mathbf{v}_2 + 3\mathbf{v}_3$, which when rearranged gives

$$\mathbf{v}_1 - 4\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}.$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

[5 pts] (c) Determine, with proof, whether the set $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent or linearly independent.

Solution. We have to look for solutions to the equation $\alpha\mathbf{v}_2 + \beta\mathbf{v}_3 + \gamma\mathbf{v}_= \mathbf{0}$, i.e., to the equation

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This vector equation is equivalent to the system of linear equations

$$\begin{array}{ccccccc} \alpha & & & + & 2\gamma & = & 0 \\ & & \beta & & & = & 0 \\ \alpha & - & \beta & + & 3\gamma & = & 0. \end{array}$$

The second equation gives $\underline{\beta = 0}$. Subtracting the first equation from the third gives $-\beta + \gamma = 0$ and since we know that $\beta = 0$, we obtain $\underline{\gamma = 0}$. Plugging back this value of γ into the first equation yields $\underline{\alpha = 0}$. Thus the system has the unique solution $\alpha = 0$, $\beta = 0$, $\gamma = 0$, which means that $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.