STUDENT NUMBER:

APSC 174 — Midterm 1

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Solutions

Instructions: The exam has **five** questions, worth a total of 100 marks.

Answer **all questions**, writing clearly in the space provided, including the provided space for additional work. If you need more room, continue to answer on the back of the **previous page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order.

No textbook, lecture notes, calculator, computer, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	Total
/20	/20	/15	/20	/25	/100

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1. In the vector space $\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$ under the usual addition and scalar multiplication operations seen in class, consider the vectors $\mathbf{w} = (3, 6, 7)$, $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (1, 1, 1)$, and $\mathbf{v}_3 = (0, 0, 1)$.

[8 pts] (a) Is $\mathbf{w}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? (Justify your answer.)

Solution. A linear combination of $\mathbf{v}_1$,$\mathbf{v}_2$,$\mathbf{v}_3$ is any vector of the form

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 \text{ with } x_1, x_2, x_3 \in \mathbb{R}.$$

Setting up an equation to determine if such x_1, x_2, x_3 exist so that the linear combination equals the vector $\mathbf{w}$, we obtain:

$$x_1 \begin{pmatrix}1, & 2, & 0\end{pmatrix} + x_2 \begin{pmatrix}1, & 1, & 1\end{pmatrix} + x_3 \begin{pmatrix}0, & 0, & 1\end{pmatrix} = \begin{pmatrix}3, & 6, & 7\end{pmatrix}$$

Matching component by component, we get the following scalar equations:

$$x_1 + x_2 = 3 \tag{1}$$
$$2x_1 + x_2 = 6 \tag{2}$$
$$x_2 + x_3 = 7 \tag{3}$$

Equation $(2) - (1)$ gives $x_1 = 3$.
Using that with (2) gives $x_2 = 6 - 2(3) = 0$.
Then (3) becomes $x_3 = 7$.
Since we were able to find the values $x_1 = 3, x_2 = 0, x_3 = 7$ that satisfied the equation, then we have that $\mathbf{w}$ *is* a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

[6 pts] (b) Is the set $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent, or linearly independent? (Justify your answer.)

Solution. This set of vectors is linearly independent if the only coefficients $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ which satisfy $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{w} = \mathbf{0}$ are all 0 values.
To test this, we set up the system of equations:

$$\alpha_1 \begin{pmatrix}1, & 2, & 0\end{pmatrix} + \alpha_2 \begin{pmatrix}1, & 1, & 1\end{pmatrix} + \alpha_3 \begin{pmatrix}3, & 6, & 7\end{pmatrix} = \begin{pmatrix}0, & 0, & 0\end{pmatrix}$$

Matching component by component, we get the following scalar equations:

$$\alpha_1 + \alpha_2 + 3\alpha_3 = 0 \tag{4}$$
$$2\alpha_1 + \alpha_2 + 6\alpha_3 = 0 \tag{5}$$
$$\alpha_2 + 7\alpha_3 = 0 \tag{6}$$

Equation $(5) - 2 \times (4)$ gives $-\alpha_2 = 0$ or simply $\alpha_2 = 0$.
With that value determined, then (6) gives $0 + 7\alpha_3 = 0$, so $\alpha_3 = 0$ too.
Finally, (4) becomes $\alpha_1 + 0 + 3(0) = 0$, so α_1 must equal 0 as well.
Thus the only solution to $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{w} = \mathbf{0}$ is for $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which means that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$ is *linearly independent*.

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[6 pts] (c) Is the set $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_3\}$ linearly dependent, or linearly independent? (Justify your answer.)

Solution.
Short option: We saw in part (a) that $3\mathbf{v}_1 + 0\mathbf{v}_2 + 7\mathbf{v}_3 = \mathbf{w}$. Rewriting this into the form for the linear independence equation, we have:

$$3\mathbf{v}_1 + 7\mathbf{v}_3 - \mathbf{w} = \mathbf{0}$$

Since the scalar coefficients are *not* all zero, we conclude that the set $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{w}\}$ is *linearly dependent*.

Longer option: The set of vectors $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{w}$ is linearly independent if the only coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ which satisfy $\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_3 + \beta_3\mathbf{w} = \mathbf{0}$ are all 0 values.
To test this, we set up the system of equations:

$$\beta_1 \begin{pmatrix}1, & 2, & 0\end{pmatrix} + \beta_2 \begin{pmatrix}0, & 0, & 1\end{pmatrix} + \beta_3 \begin{pmatrix}3, & 6, & 7\end{pmatrix} = \begin{pmatrix}0, & 0, & 0\end{pmatrix}$$

Matching component by component, we get the following scalar equations:

$$\beta_1 + 3\beta_3 = 0 \tag{7}$$
$$2\beta_1 + 6\beta_3 = 0 \tag{8}$$
$$\beta_2 + 7\beta_3 = 0 \tag{9}$$

We note that Equation (8) is simply a two times multiple of Equation (7), so we can ignore Equation (8). This gives us the reduced system:

$$\beta_1 + 3\beta_3 = 0$$
$$\beta_2 + 7\beta_3 = 0$$

Multiple non-zero solutions to this are possible. E.g. Letting $\beta_3 = 1$, we can obtain $\beta_1 = -3$ and $\beta_2 = -7$. Thus: $-3\mathbf{v}_1 + -7\mathbf{v}_3 + \mathbf{w} = \mathbf{0}$, even with non-zero coefficients, so the set $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{w}\}$ is *linearly dependent*.

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2. Consider the vector space $\mathbb{R}^2 = \{(x, y) \ : \ x, y \in \mathbb{R}\}$ under the usual addition and scalar multiplication operations seen in class.

[6 pts] (a) Determine, with proof, whether or not the set $\mathbf{H}_1 = \{(x, y) \in \mathbb{R}^2 \ : \ y \geq x\}$ is a vector subspace of $\mathbb{R}^2$.

Solution. We need to check whether $\mathbf{H}_1$ satisfies the three conditions of a vector subspace.

(i) **Q:** Is $\mathbf{0} \in \mathbf{H}_1$?

A: Yes. In $\mathbb{R}^2$, $\mathbf{0} = (0, 0)$ and since the second component of $(0, 0)$ is greater than or equal than its first component (as clearly $0 \geq 0$), we conclude that $\mathbf{0} \in \mathbf{H}_1$.

(ii) **Q:** If $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}_1$, is $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{H}_1$?

A: Yes. Let $\mathbf{w}_1 = (x_1, y_1) \in \mathbf{H}_1$; then we know that $y_1 \geq x_1$. Similarly, let $\mathbf{w}_2 = (x_2, y_2) \in \mathbf{H}_1$; then we know that $y_2 \geq x_2$.

Then $\mathbf{w}_1 + \mathbf{w}_2 = (x_1 + x_2, \, y_1 + y_2)$, and this vector is in $\mathbf{H}_1$ since $y_1 + y_2 \geq x_1 + x_2$.

(iii) **Q:** If $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbf{H}_1$, is $\alpha \cdot \mathbf{w} \in \mathbf{H}_1$?

A: No. Here is a counterexample: Let $\alpha = -1$ and $\mathbf{w} = (1, 2) \in \mathbf{H}_1$. Then $\alpha \cdot \mathbf{w} = (\alpha x, \, \alpha y) = (-1, -2)$. But $(-1, -2) \notin \mathbf{H}_1$ since its second component is *less* than its first component. Thus $\alpha \cdot \mathbf{w} \notin \mathbf{H}_1$.

Since $\mathbf{H}_1$ does not satisfy the third condition (i.e., it is not closed under scalar multiplication), we conclude that $\mathbf{H}_1$ is *not* a vector subspace of $\mathbb{R}^2$.

[6 pts] (b) Given the set $\mathbf{H}_2 = \{(x, y) \in \mathbb{R}^2 \ : \ y \leq x\}$, determine the set $\mathbf{H}_1 \cap \mathbf{H}_2$.

Solution. Using the definition of the intersection of two sets, we have:

$$\begin{aligned}
\mathbf{H}_1 \cap \mathbf{H}_2 &= \left\{(x, y) \in \mathbb{R}^2 \ : \ (x, y) \in \mathbf{H}_1 \text{ and } (x, y) \in \mathbf{H}_2\right\} \\
&= \left\{(x, y) \in \mathbb{R}^2 \ : \ y \geq x \text{ and } y \leq x\right\} \\
&= \left\{(x, y) \in \mathbb{R}^2 \ : \ y = x\right\}.
\end{aligned}$$

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[8 pts] (c) Determine, with proof, whether or not $\mathbf{H}_1 \cap \mathbf{H}_2$ is a vector subspace of $\mathbb{R}^2$.

Solution. As in part (a) above, we need to check whether $\mathbf{H}_1 \cap \mathbf{H}_2$ satisfies the three conditions of a vector subspace.

(i) **Q:** Is $\mathbf{0} \in \mathbf{H}_1 \cap \mathbf{H}_2$?

A: Yes. The zero vector $\mathbf{0} = (0,0)$ of $\mathbb{R}^2$ clearly satisfies the property of $\mathbf{H}_1 \cap \mathbf{H}_2$ since its components both match (with each being 0). Thus $\mathbf{0} \in \mathbf{H}_1 \cap \mathbf{H}_2$.

(ii) **Q:** If $\mathbf{w}_1,\, \mathbf{w}_2 \in \mathbf{H}_1 \cap \mathbf{H}_2$, is $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{H}_1 \cap \mathbf{H}_2$?

A: Yes. Let $\mathbf{w}_1 = (x_1, y_1) \in \mathbf{H}_1 \cap \mathbf{H}_2$; then $y_1 = x_1$. Also, let $\mathbf{w}_2 = (x_2, y_2) \in \mathbf{H}_1 \cap \mathbf{H}_2$; then $y_2 = x_2$.

Thus $\mathbf{w}_1 + \mathbf{w}_2 = (x_1 + x_2,\, y_1 + y_2)$, and this vector is in $\mathbf{H}_1 \cap \mathbf{H}_2$ since $y_1 + y_2 = x_1 + x_2$.

(iii) **Q:** If $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbf{H}_1$, is $\alpha \cdot \mathbf{w} \in \mathbf{H}_1 \cap \mathbf{H}_2$?

A: Yes. Let $\alpha \in \mathbb{R}$ and $\mathbf{w} = (x, y) \in \mathbf{H}_1 \cap \mathbf{H}_2$; then $y = x$. Hence $\alpha \cdot \mathbf{w} = (\alpha x,\, \alpha y) = (\alpha x,\, \alpha x) \in \mathbf{H}_1 \cap \mathbf{H}_2$ (as both components of $\alpha \cdot \mathbf{w}$ are identical). Thus $\alpha \cdot \mathbf{w} \in \mathbf{H}_1 \cap \mathbf{H}_2$.

As $\mathbf{H}_1 \cap \mathbf{H}_2$ satisfies all three conditions above, we conclude that $\mathbf{H}_1 \cap \mathbf{H}_2$ $\underline{\text{is a vector subspace of } \mathbb{R}^2}$.

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3. Recall that $C^\infty(\mathbb{R})$ is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times. The operations on $C^\infty(\mathbb{R})$ are the usual addition and scalar multiplication of functions as seen in class. Let

$$\mathbf{W} = \left\{ f \in C^\infty(\mathbb{R}) \ : \ f''(0) + f'(\pi/2) = 0 \right\} \subset C^\infty(\mathbb{R}).$$

For instance, $\sin(x) \in \mathbf{W}$ since $\sin(x)' = \cos(x)$, $\sin(x)'' = -\sin(x)$, and so

$$\sin(0)'' + \sin(\pi/2)' = -\sin(0) + \cos(\pi/2) = -0 + 0 = 0.$$

[5 pts] (a) Is $\cos(x) \in \mathbf{W}$?

Solution. No. The function $\cos(x)$ has derivatives of arbitrary order, so $\cos(x) \in C^\infty(\mathbb{R})$. However, $\cos(x)' = -\sin(x)$ and $\cos(x)'' = -\cos(x)$, and so

$$\cos(0)'' + \cos(\pi/2)' = -\cos(0) - \sin(\pi/2) = -1 - 1 = -2 \neq 0,$$

which implies that $\underline{\cos(x) \notin \mathbf{W}}$.

[10 pts] (b) Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

Solution. We check the three conditions for $\mathbf{W}$ to be a subspace:

(i) **Q:** Is $\mathbf{0} \in \mathbf{W}$?

A: Yes. Since the zero vector $\mathbf{0}$ in $\mathbf{W}$ is the all zero function $f(x) = 0$ for all $x \in \mathbb{R}$ and since $f'(x) = 0$ and $f''(x) = 0$ for all x, we have

$$f''(0) + f'(\pi/2) = 0 + 0 = 0,$$

we indeed have $\underline{\mathbf{0} \in \mathbf{W}}$.

(ii) **Q:** If $f_1, f_2 \in \mathbf{W}$, is $f_1 + f_2 \in \mathbf{W}$?

A: Yes. Since $f_1, f_2 \in \mathbf{W}$, we know that $f_1''(0) + f_1'(\pi/2) = 0$ and $f_2''(0) + f_2'(\pi/2) = 0$, and so

$$\begin{aligned} (f_1 + f_2)''(0) + (f_1 + f_2)'(\pi/2) &= \big(f_1''(0) + f_2''(0)\big) + \big(f_1'(\pi/2) + f_2'(\pi/2)\big) \\ &= f_1''(0) + f_1'(\pi/2) + f_2''(0) + f_2'(\pi/2) \\ &= 0. \end{aligned}$$

Thus $\underline{f_1 + f_2 \in \mathbf{W}.}$

(iii) **Q:** If $\alpha \in \mathbb{R}$ and $f \in \mathbf{W}$, is $\alpha f \in \mathbf{W}$?

A: Yes. Since $f \in \mathbf{W}$ implies $f''(0) + f'(\pi/2) = 0$, we have

$$(\alpha f)''(0) + (\alpha f)'(\pi/2) = \alpha f''(0) + \alpha f'(\pi/2) = \alpha\big(f''(0) + f'(\pi/2)\big) = 0,$$

which means that $\underline{\alpha f \in \mathbf{W}.}$

Since $\mathbf{W}$ satisfies all three conditions, $\underline{\mathbf{W} \text{ is a subspace}}$.

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4. Consider a vector space $\mathbf{V}$. Answer the questions below about this vector space.

[5 pts] (a) Define what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$ to be *linearly dependent*.

Solution. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

for coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ has solutions other than $\alpha_1 = 0$, $\alpha_2 = 0$, $\ldots \alpha_n = 0$ (here $\mathbf{0}$ is the zero vector in $\mathbf{V}$). In other words, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent if the zero vector can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with coefficients that are not all zero.

[5 pts] (b) Let $\mathbf{0}$ be the zero vector in $\mathbf{V}$ and let $\mathbf{v} \in \mathbf{V}$ be an arbitrary non-zero vector. Determine, with proof, whether the set $\{\mathbf{0}, \mathbf{v}\}$ is linearly dependent or independent.

Solution.
Argument 1: According to part (a), we have to look at all the solutions of

$$\alpha_1 \mathbf{0} + \alpha_2 \mathbf{v} = \mathbf{0}.$$

Since $\alpha_1 \mathbf{0} = \mathbf{0}$ for all $\alpha_1 \in \mathbb{R}$, if we pick and arbitrary $\alpha_1 \neq 0$ and let $\alpha_2 = 0$, the above equation will hold. Thus $\mathbf{0}$ can be written as a linear combination of $\{\mathbf{0}, \mathbf{v}\}$ with coefficients that are not all zero, implying that the set $\{\mathbf{0}, \mathbf{v}\}$ is linearly dependent.

Argument 2: We have proved in class that a set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if either $\mathbf{v}_1 = \alpha \mathbf{v}_2$ or $\mathbf{v}_2 = \beta \mathbf{v}_1$ for some $\alpha, \beta \in \mathbb{R}$, i.e., if and only if one of the vectors is a scalar multiple of the other one. Here we have $0 \cdot \mathbf{v} = \mathbf{0}$, so $\mathbf{0}$ is a scalar multiple of $\mathbf{v}$, which implies that the set $\{\mathbf{0}, \mathbf{v}\}$ is linearly dependent.

[5 pts] (c) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbf{V}$. Define what it means for a vector $\mathbf{v} \in \mathbf{V}$ to be in the span $S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$.

Solution. The vector $\mathbf{v} \in \mathbf{V}$ is in the span $S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$ if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n.$$

In other words, $\mathbf{v} \in S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$ if $\mathbf{v}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

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[5 pts] (d) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbf{V}$ and assume that $\mathbf{v} \in S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$. Is the set $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ linearly dependent or independent? Justify (i.e., provide an argument for, or prove) your answer.

Solution. From part (c), $\mathbf{v}$ is in the span $S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$ if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n.$$

But then we have

$$\begin{aligned} \mathbf{0} &= -\mathbf{v} + \mathbf{v} \\ &= (-1)\mathbf{v} + \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n, \end{aligned}$$

so $\mathbf{0}$ can be written as linear combination of $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that at last one of the coefficients (namely, that of $\mathbf{v}$) is nonzero. Thus the set $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is <u>linearly dependent</u>.

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5. Consider the set $\mathbf{V} = \{(x, y, z) \,:\, x, y, z \in \mathbb{R},\ z > 0\}$ with the following **new** addition and scalar multiplication operations:

Addition: For any (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbf{V}$,

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2,\ y_1 + y_2 - 3,\ z_1 z_2).$$

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $(x, y, z) \in \mathbf{V}$,

$$\alpha \cdot (x, y, z) = (\alpha x,\ \alpha y - 3\alpha + 3,\ z^{\alpha}).$$

It can be proved (and you do not have to do this) that $\mathbf{V}$ with these operations is a vector space.

[5 pts] (a) Determine $2 \cdot \big((-5, -3, 2) + (6, 4, 3)\big)$ using the operations in $\mathbf{V}$.

Solution. Using the operations of addition and multiplications in $\mathbf{V}$ defined above, we have:

$$\begin{aligned} 2 \cdot \big((-5, -3, 2) + (6, 4, 3)\big) &= 2 \cdot \big(-5 + 6,\ -3 + 4 - 3,\ (2)(3)\big) \\ &= 2 \cdot \big(1,\ -2,\ 6\big) \\ &= \big((2)(1),\ (2)(-2) - 3(2) + 3,\ 6^2\big) \\ &= \underline{\big(2,\ -7,\ 36\big)}. \end{aligned}$$

[5 pts] (b) Determine the zero vector $\mathbf{0}$ of $\mathbf{V}$.

Solution.
Argument 1: We have learned in class that if $\mathbf{v}$ is any vector in a vector space $\mathbf{V}$, then $0 \cdot \mathbf{v} = \mathbf{0}$. Hence for any vector $\mathbf{v} = (x, y, z) \in \mathbf{V}$, we have

$$\mathbf{0} = 0 \cdot \mathbf{v} = 0 \cdot (x, y, z) = \big((0)(x),\ (0)(y) - 3(0) + 3,\ z^0\big) = (0, 3, 1).$$

Hence $\underline{\mathbf{0} = (0, 3, 1)}$.

Argument 2: We can also determine $\mathbf{0}$ by the requirement (from the Axioms) that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for any vector $\mathbf{v} \in \mathbf{V}$. Letting $\mathbf{v} = (x, y, z) \in \mathbf{V}$ be arbitrary and denoting the components of $\mathbf{0}$ by x', y' and z' (so that $\mathbf{0} = (x', y', z')$), we obtain the equation

$$(x, y, z) + (x', y', z') = (x, y, z)$$

or, using the addition operation in $\mathbf{V}$,

$$\big(x + x',\ y + y' - 3,\ zz'\big) = (x, y, z).$$

Equating the first, second, and third components of the two vectors, we respectively get $x + x' = x$, implying that $x' = 0$, $y + y' - 3 = y$, implying that $y' = 3$, and $zz' = z$, implying that $z = 1$ (as both z and z' are positive). Thus $\underline{\mathbf{0} = (0, 3, 1)}$.

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[5 pts] (c) Given the vector $(x,y,z) \in \mathbf{V}$, determine its additive inverse; that is, find a vector (x',y',z') such that $(x,y,z)+(x',y',z')=\mathbf{0}$.

Solution.
Argument 1: Here we can use the fact, learned in class, that the additive inverse of any vector $\mathbf{v} \in \mathbf{V}$ is $(-1)\cdot\mathbf{v}$. Thus the additive inverse of $\mathbf{v}=(x,y,z) \in \mathbf{V}$ is given by

$$-\mathbf{v} = (-1)\cdot(x,y,z) = \big((-1)(x),\ (-1)(y)-3(-1)+3,\ z^{-1}\big) = \big(-x,\ -y+6,\ 1/z\big).$$

Hence the additive inverse of (x,y,z) is $\underline{\big(-x,\ -y+6,\ 1/z\big)}$.

Argument 2:
Alternatively, we can find the additive inverse (x',y',z') of $(x,y,z)\in\mathbf{V}$ from the requirement that $(x,y,z)+(x',y',z')=\mathbf{0}$. We have

$$(x,y,z)+(x',y',z') = (x+x', y+y'-3, zz')$$

and since $\mathbf{0}=(0,3,1)$, we obtain the vector equation

$$(x+x', y+y'-3, zz') = (0,3,1)$$

i.e., $x+x'=0$, $y+y'-3=3$ and $zz'=1$. Solving for x', y' and z' yields $x'=-x$, $y'=-y+6$ and $z'=1/z$. Thus $(x',y',z')=\underline{\big(-x,\ -y+6,\ 1/z\big)}$.

[5 pts] (d) Given $\mathbf{w}_1=(0,3,1)$, $\mathbf{w}_2=(-1,2,3)$ and $\mathbf{w}_3=(2,2,9)$ in $\mathbf{V}$, determine (using the operations in $\mathbf{V}$) whether or not $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Solution. We need to verify (using the operations in $\mathbf{V}$) whether there exist scalars α_1 and α_2 in $\mathbb{R}$ such that $\mathbf{w}_3=\alpha_1\cdot\mathbf{w}_1+\alpha_2\cdot\mathbf{w}_2$. This equation is as follows:

$$(2,2,9) = \alpha_1\cdot(0,3,1)+\alpha_2\cdot(-1,2,3).$$

But noting that $\mathbf{w}_1=(0,3,1)=\mathbf{0}$ is the zero vector of $\mathbf{V}$ (as seen in part (b)), we have $\alpha_1\cdot(0,3,1)=\alpha_1\cdot\mathbf{0}=\mathbf{0}$ (for any $\alpha_1\in\mathbb{R}$) by the property of the zero vector. Also, $\mathbf{0}+\alpha_2\cdot(-1,2,3)=\alpha_2\cdot(-1,2,3)$ by the property of the zero vector. Thus the above equation simplifies to:

$$(2,2,9) = \alpha_2\cdot(-1,2,3) = \big(-\alpha_2,\ \alpha_2(2)-3(\alpha_2)+3,\ 3^{\alpha_2}\big)$$

or equivalently,

$$(2,2,9) = \big(-\alpha_2,\ -\alpha_2+3,\ 3^{\alpha_2}\big).$$

Equating the components of the above two vectors yields the following simultaneous equations in α_2:

$$\begin{cases} 2=-\alpha_2 \\ 2=-\alpha_2+3 \\ 9=3^{\alpha_2} \end{cases}$$

Solving for α_2 in the above three equations result in *contradictory* solutions: $\alpha_2=-2$ in the first equation, $\alpha_2=1$ in the second equation and $\alpha_2=2$ in the third equation. Thus there does not exist a valid choice for the pair of scalars (α_1,α_2) such that $\mathbf{w}_3=\alpha_1\cdot\mathbf{w}_1+\alpha_2\cdot\mathbf{w}_2$. Thus $\mathbf{w}_3$ is *not* a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

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[5 pts] (e) Determine, with proof (using the operations in $\mathbf{V}$), whether or not the set of the above vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

Solution. Since $\mathbf{w}_1 = \mathbf{0}$ is the zero vector of $\mathbf{V}$ (as noted in part (d)), we directly obtain that the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is *not linearly independent*: indeed choosing scalars $\alpha_1 = 4 \neq 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$ (i.e, not all zero) yields

$$\alpha_1 \cdot \mathbf{w}_1 + \alpha_2 \cdot \mathbf{w}_2 + \alpha_3 \cdot \mathbf{w}_3 = 4 \cdot \mathbf{0} + 0 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3 = \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is *not* linearly independent, i.e., it is linearly dependent.