APSC 174 – Midterm 1

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Solutions

Instructions:

The exam has **five** questions, worth a total of 100 marks.

Separately write on paper your answers to each problem. At the end of the test, scan and upload your answers to each problem/question in their corresponding slot on **Crowdmark**.

To receive full credit you must show your work, clearly and in order.

Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	Total
/20	/20	/20	/20	/20	/100

1. In the vector space

$$\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class, consider the vectors $\mathbf{v}_1 = (2, 1, 0)$, $\mathbf{v}_2 = (1, 2, 0)$, $\mathbf{v}_3 = (4, -1, 0)$.

[8 pts] (a) Is $\mathbf{v}_3$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? (Justify your answer.)

Solution. Asking if $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ is equivalent to asking if there exist $x_1, x_2 \in \mathbb{R}$ such that the following system of equations has a solution:

$$x_1 \begin{pmatrix} 2, & 1, & 0 \end{pmatrix} + x_2 \begin{pmatrix} 1, & 2, & 0 \end{pmatrix} = \begin{pmatrix} 4, & -1, & 0 \end{pmatrix}$$

Matching component by component, we get the following scalar equations:

$$2x_1 + x_2 = 4 \tag{1}$$
$$x_1 + 2x_2 = -1 \tag{2}$$

Multiplying the second equation by -2 and adding up the two equations gives that $-3x_2 = 6$ or $x_2 = -2$. Therefore, by substituting $x_2 = -2$ in either of the two equations (for example, in the equation $2x_1 + x_2 = 4$), we get that $2x_1 - 2 = 4$ and hence $x_1 = 3$. Since the above system does indeed have a solution, we conclude that $\mathbf{v}_3$ is in a linear combination of $\mathbf{v}_1, \mathbf{v}_2$. Namely, we have $\mathbf{v}_3 = 3\mathbf{v}_1 - 2\mathbf{v}_2$.

[6 pts] (b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent, or linearly independent? (Justify your answer.)

Solution. For the special case of testing the linear independence of a pair of vectors (rather than 3 or more vectors), it suffices to check if one of the vectors is a scalar multiple of the other. Here, $\mathbf{v}_1, \mathbf{v}_2$ are clearly not multiples of one another. Namely, if $\mathbf{v}_2 = \alpha \mathbf{v}_1$, then we must have $1 = \alpha \cdot 2$ and $2 = \alpha \cdot 1$ which is impossible. Therefore, we conclude that $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent.

[6 pts] (c) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent, or linearly independent? (Justify your answer.)

Solution. Using part (a), we have $3\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. By the definition of linear dependence, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Alternatively, using the theorem stating " a collection of vectors is linearly dependent if and only if one of them can be written as a linear combination of the others", since $\mathbf{v}_3 = 3\mathbf{v}_1 - 2\mathbf{v}_2$ by part (a), we conclude that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

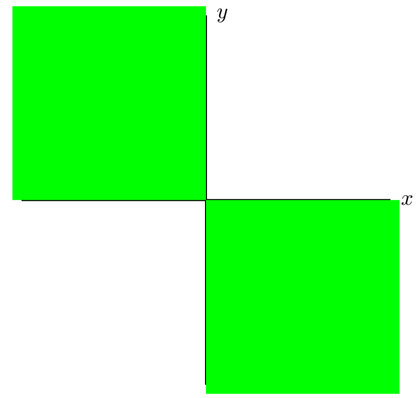
2. Consider the vector space

$$\mathbb{R}^2 = \{(x, y) \ : \ x, y \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class.

Let $\mathbf{H}$ be the set of all points in the second and fourth quadrants of $\mathbb{R}^2$, as indicated by the *green shaded regions* in the diagram at right. More precisely, we have that

$$\mathbf{H} = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq 0\}.$$



$\mathbb{R}^2$, with the set $\mathbf{H}$ shaded in green.

[4 pts] (a) Does $\mathbf{H}$ contain the zero vector of $\mathbb{R}^2$? Justify your answer.

Solution. Yes, the set $\mathbf{H}$ does contain the zero vector $(0, 0)$ as $0 \cdot 0 = 0 \leqslant 0$.

[6 pts] (b) Is $\mathbf{H}$ closed under addition? If yes, prove your statement; if not, provide a counter-example.

Solution. Notice that the vectors $\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (0, 1)$ are both in $\mathbf{H}$ as $0 \cdot 1 = 1 \cdot 0 = 0 \leqslant 0$. However, $\mathbf{v}_1 + \mathbf{v}_2 = (1, 0) + (0, 1) = (1, 1)$ is not in $\mathbf{H}$ as $1 \cdot 1 = 1 > 0$. Hence, $\mathbf{H}$ is not closed under addition.

[6 pts] (c) Is $\mathbf{H}$ closed under scalar multiplication? If yes, prove your statement; if not, provide a counter-example.

Solution. Let $(x, y) \in \mathbf{H}$ and let $\alpha \in \mathbb{R}$. Since $(x, y) \in \mathbf{H}$, we have $x \cdot y \leqslant 0$. Notice that since α^2 is non-negative, we also have $\alpha^2 xy \leqslant 0$. Observe that $\alpha \cdot (x, y) = (\alpha x, \alpha y)$. Furthermore, we have $(\alpha x)(\alpha y) = \alpha^2 x \cdot y \leqslant 0$. Hence, $\mathbf{H}$ is closed under scalar multiplication.

[4 pts] (d) Determine whether or not $\mathbf{H}$ is a vector subspace of $\mathbb{R}^2$, referring to your answers in parts (a)-(c).

Solution. Using part *(b)*, since $\mathbf{H}$ is not closed under addition, we conclude that $\mathbf{H}$ is not a subspace of $\mathbb{R}^2$.

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3. Recall that $C^\infty(\mathbb{R})$ is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times. The operations on $C^\infty(\mathbb{R})$ are the usual addition and scalar multiplication of functions as seen in class. Let

$$\mathbf{W} = \left\{ f \in C^\infty(\mathbb{R}) \ : \ f'(0) = f(3) \right\} \subset C^\infty(\mathbb{R})$$

where f' denotes the derivative of f.

[8 pts] (a) Consider the functions f_1 and f_2 in $C^\infty(\mathbb{R})$ given by

$$f_1(x) = -1 - 4x + x^2$$

and

$$f_2(x) = (x-3)^2$$

for $x \in \mathbb{R}$. Determine (with justification) whether the functions f_1 and/or f_2 belong to $\mathbf{W}$.

Solution.
For f_1, $f_1'(x) = -4 + 2x$. Hence, $f_1'(0) = -4$. Furthermore, $f_1(3) = -1 - 4(3) + (3)^2 = -1 - 12 + 9 = -4$. Since $f_1'(0) = f_1(3) = -4$, we conclude that $f_1 \in \mathbf{W}$.
For f_2, we have $f_2'(x) = 2(x-3)$ which implies that $f_2'(0) = 2(-3) = -6$. However, we have $f_2(3) = (3-3)^2 = 0$. Since $f_2'(0) \neq f_2(3)$, the function f_2 is not in $\mathbf{W}$.

[12 pts] (b) Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

Solution.

1. First, observe that the zero function $f(x) = 0$ is in $\mathbf{W}$ as $f'(0) = f(3) = 0$.

2. Let $f, g \in \mathbf{W}$. This implies that $f'(0) = f(3)$ and $g'(0) = g(3)$. Now, recall that $(f+g)(x) = f(x) + g(x)$ and $(f+g)'(x) = f'(x) + g'(x)$. Therefore, we have $(f+g)'(0) = f'(0) + g'(0) = f(3) + g(3) = (f+g)(3)$. Hence, we conclude that $\mathbf{W}$ is closed under addition.

3. Let $\alpha \in \mathbb{R}$ and let $f \in \mathbf{W}$. Since $f \in \mathbf{W}$, we have $f'(0) = f(3)$. Now, recall that $(\alpha f)(x) = \alpha \cdot f(x)$ and $(\alpha f)'(x) = \alpha \cdot f'(x)$. Therefore, we have $(\alpha f)'(0) = \alpha \cdot f'(0) = \alpha \cdot f(3) = (\alpha f)(3)$. Hence, we conclude that $\mathbf{W}$ is closed under scalar multiplication.

In light of $1, 2$ and 3 above, we conclude that $\mathbf{W}$ is a subspace.

4. Consider a vector space $\mathbf{V}$. Answer the questions below about this vector space.

[5 pts] (a) Define what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$ to be *linearly independent.*

Solution. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$ is linearly independent if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, i.e., if the zero vector $\mathbf{0}$ in $\mathbf{V}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ only with scalar coefficients that are all zero.

[5 pts] (b) Define the span $S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)}$ of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$.

Solution. The span $S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)}$ of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$ is the set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, i.e.,

$$S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\}.$$

[5 pts] (c) Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ in $\mathbf{V}$, show that if every vector $\mathbf{w}$ in $S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p)}$ can be written in *exactly one way* as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent.

Solution.
Argument 1: By a theorem learned in class, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a linearly dependent set, then every vector $\mathbf{w}$ in $S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p)}$ can be written infinitely many ways as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$. Thus if every vector $\mathbf{w}$ in $S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p)}$ can be written in exactly one way as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ cannot be linearly dependent, so it is *linearly independent.*

Argument 2: We first note that $\mathbf{0} \in S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p)}$ since

$$\mathbf{0} = 0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + \cdots + 0\cdot\mathbf{v}_p. \qquad (*)$$

If every vector $\mathbf{w}$ in $S_{(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p)}$ can be written in exactly one way as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$, then $\mathbf{0}$ can only be written in exactly one way as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$. Therefore, if

$$\mathbf{0} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p,$$

then by equation $(*)$ we must have $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent.

[5 pts] (d) Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors in $\mathbf{V}$. Fix $\mathbf{u} \in \mathbf{V}$ and let $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{u}$ and $\mathbf{u}_2 = \mathbf{v}_2 + \mathbf{u}$. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set, then we must have that $\mathbf{u} \in S_{(\mathbf{v}_1,\mathbf{v}_2)}$.

Solution. If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set, then there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \mathbf{0}$ and at least one of α and β is not equal to zero. The vector equation $\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \mathbf{0}$ is equivalent to

$$\alpha(\mathbf{v}_1 + \mathbf{u}) + \beta(\mathbf{v}_2 + \mathbf{u}) = \mathbf{0}$$

i.e.,

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = -(\alpha + \beta)\mathbf{u}.$$

We note that $\alpha + \beta \neq 0$ since otherwise the above vector equation would give $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{0}$ with at least one of α and β not being zero, which would contradict the assumption that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set. Thus we can divide both sides by $-(\alpha + \beta)$ to obtain

$$\left(\frac{-\alpha}{\alpha+\beta}\right)\mathbf{v}_1 + \left(\frac{-\beta}{\alpha+\beta}\right)\mathbf{v}_2 = \mathbf{u}$$

proving that $\mathbf{u} \in S_{(\mathbf{v}_1,\mathbf{v}_2)}$.

5. Consider the set $\mathbf{V} = \{x \in \mathbb{R} : x > 3\}$ with the following **new** addition and scalar multiplication operations:

Addition: For any $x, y \in \mathbf{V}$,

$$x \oplus y = xy - 3(x + y) + 12.$$

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $x \in \mathbf{V}$,

$$\alpha \cdot x = (x - 3)^{\alpha} + 3.$$

It can be proved (and you do not have to do this) that $\mathbf{V}$ with these operations is a vector space.

[4 pts] (a) Determine $4 \oplus 5$ and $-2 \cdot 5$ using the operations in $\mathbf{V}$.

Solution. Using the operations of addition and multiplications in $\mathbf{V}$ defined above, we have:

$$\begin{aligned} 4 \oplus 5 &= 4 \times 5 - 3(4 + 5) + 12 \\ &= 20 - 27 + 12 = 5 \end{aligned}$$

and

$$-2 \cdot 5 = (5 - 3)^{-2} + 3 = \frac{1}{4} + 3 = 3.25$$

[4 pts] (b) Determine the zero vector $\mathbf{0}$ of $\mathbf{V}$.

Solution.
<u>Argument 1:</u> We have learned in class that if $\mathbf{v}$ is any vector in a vector space $\mathbf{V}$, then $0 \cdot \mathbf{v} = \mathbf{0}$. Thus for any $x \in \mathbf{V}$, we have

$$\mathbf{0} = 0 \cdot x = (x - 3)^0 + 3 = 1 + 3 = 4$$

Hence $\mathbf{0} = 4$.

<u>Argument 2:</u> We can also determine $\mathbf{0}$ by the requirement (from the Axioms) that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for any vector $\mathbf{v} \in \mathbf{V}$. Letting $\mathbf{v} = x \in \mathbf{V}$ be arbitrary and denoting $\mathbf{0}$ by z, we obtain the equation

$$x \oplus z = x$$

or, using the addition operation in $\mathbf{V}$,

$$xz - 3(x + z) + 12 = x.$$

Rearranging this, we get

$$z(x - 3) = 4x - 12$$

Since $x > 3$, we can divide both sides by $x - 3$ to obtain $z = \mathbf{0} = 4$.

[4 pts] (c) Given $x \in \mathbf{V}$, determine its additive inverse; that is, find $y \in \mathbf{V}$ such that $x \oplus y = \mathbf{0}$.

Solution.
Argument 1: Here we can use the fact, learned in class, that the additive inverse of any vector $\mathbf{v} \in \mathbf{V}$ is $(-1) \cdot \mathbf{v}$. Thus the additive inverse of $\mathbf{v} = x \in \mathbf{V}$ is given by

$$(-1) \cdot x = (x-3)^{-1} + 3 = \frac{1}{x-3} + 3.$$

Hence the additive inverse of x is $\frac{1}{x-3} + 3$.

Argument 2:
Alternatively, we can find the additive inverse y of $x \in \mathbf{V}$ from the requirement that $x \oplus y = \mathbf{0}$. We have

$$x \oplus y = xy - 3(x+y) + 12$$

and since $\mathbf{0} = 4$, we obtain the equation

$$xy - 3(x+y) + 12 = 4$$

i.e.,

$$y(x-3) = 3x - 8 = 3(x-3) + 1.$$

Dividing both sides by $x - 3 > 0$ we obtain that the additive inverse of x is $y = \frac{1}{x-3} + 3$.

[4 pts] (d) Given $u = 4$, $v = 5$ and $w = 7$ in $\mathbf{V}$, determine (using the operations in $\mathbf{V}$) whether or not w is a linear combination of u and v.

Solution. We have to decide if there exist scalars $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \cdot u \oplus \beta \cdot v = w.$$

We know from part (b) that 4 is the zero vector $\mathbf{0}$ in $\mathbf{V}$ and we know from class that $\alpha \cdot \mathbf{0} = \mathbf{0}$ for all scalars $\alpha \in \mathbb{R}$. Also, $\mathbf{0} \oplus \beta \cdot v = \beta \cdot v$. Thus the equation above reduces to

$$\beta \cdot v = w.$$

and we have to decide if there exist $\beta \in \mathbb{R}$ such that this last equation holds. Using the definition of scalar multiplication in $\mathbf{V}$ and substituting $v = 5$ and $w = 7$, we obtain

$$(5-3)^{\beta} + 3 = 7$$

i.e.

$$2^{\beta} = 4$$

from which we get $\beta = 2$. Thus $w = 2 \cdot v$ (w is a scalar multiple of v) and therefore w is a linear combination of u and v with coefficients α (arbitrary) and $\beta = 2$.

[4 pts] (e) Determine, with proof (using the operations in $\mathbf{V}$), whether or not the set $\{u, v, w\}$ above is linearly independent.

Solution. We know from part (d) that w is a scalar multiple of v. Therefore $\{v, w\}$ is not a linearly independent set. Since $\{v, w\} \subset \{u, v, w\}$ this implies (say by Tutorial 4, Problem 4(a)) that the set $\{u, v, w\}$ is not linearly independent either.