APSC 174 – Midterm 1

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A. Ableson F. Alajaji T. Linder A. Zalloum

Instructions:

The exam has **five** questions, worth a total of 100 marks.

Separately write on paper your answers to each problem. At the end of the test, scan and upload your answers to each problem/question in their corresponding slot on **Crowdmark**.

To receive full credit you must show your work, clearly and in order.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	Total
/20	/20	/20	/20	/20	/100

1. In the vector space

$$\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class, consider the vectors $\mathbf{v}_1 = (2, 1, 0)$, $\mathbf{v}_2 = (1, 2, 0)$, $\mathbf{v}_3 = (4, -1, 0)$.

[8 pts] (a) Is $\mathbf{v}_3$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? (Justify your answer.)

[6 pts] (b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent, or linearly independent? (Justify your answer.)

[6 pts] (c) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent, or linearly independent? (Justify your answer.)

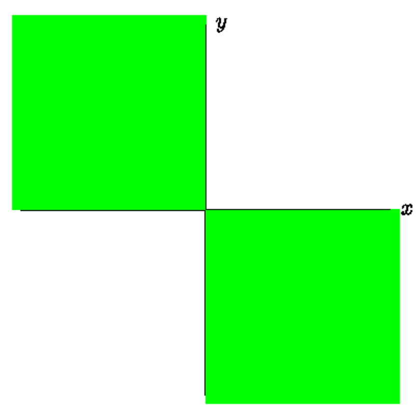
2. Consider the vector space

$$\mathbb{R}^2 = \{(x, y) \ : \ x, y \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class.

Let $\mathbf{H}$ be the set of all points in the second and fourth quadrants of $\mathbb{R}^2$, as indicated by the *green shaded regions* in the diagram at right. More precisely, we have that

$$\mathbf{H} = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq 0\}.$$



$\mathbb{R}^2$, with the set $\mathbf{H}$ shaded in green.

[4 pts] (a) Does $\mathbf{H}$ contain the zero vector of $\mathbb{R}^2$? Justify your answer.

[6 pts] (b) Is $\mathbf{H}$ closed under addition? If yes, prove your statement; if not, provide a counter-example.

[6 pts] (c) Is $\mathbf{H}$ closed under scalar multiplication? If yes, prove your statement; if not, provide a counter-example.

[4 pts] (d) Determine whether or not $\mathbf{H}$ is a vector subspace of $\mathbb{R}^2$, referring to your answers in parts (a)-(c).

3. Recall that $C^\infty(\mathbb{R})$ is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times. The operations on $C^\infty(\mathbb{R})$ are the usual addition and scalar multiplication of functions as seen in class. Let

$$\mathbf{W} = \{ f \in C^\infty(\mathbb{R}) \ : \ f'(0) = f(3) \} \subset C^\infty(\mathbb{R})$$

where f' denotes the derivative of f.

[8 pts] (a) Consider the functions f_1 and f_2 in $C^\infty(\mathbb{R})$ given by

$$f_1(x) = -1 - 4x + x^2$$

and

$$f_2(x) = (x-3)^2$$

for $x \in \mathbb{R}$. Determine (with justification) whether the functions f_1 and/or f_2 belong to $\mathbf{W}$.

[12 pts] (b) Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

4. Consider a vector space $\mathbf{V}$. Answer the questions below about this vector space.

[5 pts] (a) Define what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$ to be *linearly independent*.

[5 pts] (b) Define the span $S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)}$ of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbf{V}$.

[5 pts] (c) Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ in $\mathbf{V}$, show that if every vector $\mathbf{w}$ in $S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p)}$ can be written in *exactly one way* as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent.

[5 pts] (d) Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors in $\mathbf{V}$. Fix $\mathbf{u} \in \mathbf{V}$ and let $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{u}$ and $\mathbf{u}_2 = \mathbf{v}_2 + \mathbf{u}$. Prove that if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set, then we must have that $\mathbf{u} \in S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

5. Consider the set $\mathbf{V} = \{x \in \mathbb{R} \, : \, x > 3\}$ with the following **new** addition and scalar multiplication operations:

Addition: For any $x,\, y \in \mathbf{V}$,

$$x \oplus y = xy - 3(x + y) + 12.$$

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $x \in \mathbf{V}$,

$$\alpha \cdot x = (x - 3)^{\alpha} + 3.$$

It can be proved (and you do not have to do this) that $\mathbf{V}$ with these operations is a vector space.

[4 pts] (a) Determine $4 \oplus 5$ and $-2 \cdot 5$ using the operations in $\mathbf{V}$.

[4 pts] (b) Determine the zero vector $\mathbf{0}$ of $\mathbf{V}$.

[4 pts] (c) Given $x \in \mathbf{V}$, determine its additive inverse; that is, find $y \in \mathbf{V}$ such that $x \oplus y = \mathbf{0}$.

[4 pts] (d) Given $u = 4$, $v = 5$ and $w = 7$ in $\mathbf{V}$, determine (using the operations in $\mathbf{V}$) whether or not w is a linear combination of u and v.

[4 pts] (e) Determine, with proof (using the operations in $\mathbf{V}$), whether or not the set $\{u, v, w\}$ above is linearly independent.