APSC 174 – Midterm 1

February 2022

A. Ableson F. Alajaji I. Dimitrov K. Zhang

Solutions

Instructions:

The exam has **five** questions, worth a total of 100 marks.

Separately write on paper your answers to each problem. At the end of the test, scan and upload your answers to each problem/question in their corresponding slot on **Crowdmark**.

To receive full credit you must show your work, clearly and in order.

Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	2	3	4	5	Total
/20	/20	/20	/20	/20	/100

---

1. In the vector space
$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$
under the usual addition and scalar multiplication operations seen in class, consider the vectors $\mathbf{v}_1 = (0, 2, 3)$, $\mathbf{v}_2 = (1, 1, 2)$, $\mathbf{v}_3 = (1, 0, 1)$, and $\mathbf{v}_4 = (3, 1, 3)$.

[6 pts] (a) Is $\mathbf{v}_3$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? (Justify your answer.)

We look for scalars a, b that satisfy the equation
$a \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2 = \mathbf{v}_3$.
Using the given vector components: $a \cdot \underbrace{(0, 2, 3)}_{\mathbf{v}_1} + b \cdot \underbrace{(1, 1, 2)}_{\mathbf{v}_2} = \underbrace{(1, 0, 1)}_{\mathbf{v}_3}$.
Separating into components, we get 3 equations:

$$\begin{aligned} & & b &= 1 \\ 2a & & +b &= 0 \\ 3a & & +2b &= 1 \end{aligned}$$

From the first equation we have $b = 1$.
From the second equation, we have $2a = -b = -1$ or $a = -1/2$.
However from the third equation, we have $3a = 1 - 2b = 1 - 2 = -1$, or $a = -1/3$.
Since we cannot find a common value of the scalar multiplier a that satisfies all the equations, that means $\mathbf{v}_3$ **is not** a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

[6 pts] (b) Is $\mathbf{v}_4$ in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$? (Justify your answer.)

$\mathbf{v}_4$ being in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ also means that $\mathbf{v}_4$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so this is the same question type as part (a).
We again try to solve for scalars a, b in the linear combination equation
$a \cdot \underbrace{(0, 2, 3)}_{\mathbf{v}_1} + b \cdot \underbrace{(1, 1, 2)}_{\mathbf{v}_2} = \underbrace{(3, 1, 3)}_{\mathbf{v}_4}$.
Separating into components, we get 3 equations:

$$\begin{aligned} & & b &= 3 \\ 2a & & +b &= 1 \\ 3a & & +2b &= 3 \end{aligned}$$

From the first equation we have $b = 3$.
From the second equation, we have $2a = 1 - b = 1 - 3 = -2$ or $a = -1$.
And from the third equation, we find $3a = 3 - 2b = 3 - 6 = -3$, or $a = -1$, which is compatible with the other equations.
This indicates that we **can** write $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:
$\underbrace{(-1)}_{a} \cdot (0, 2, 3) + \underbrace{3}_{b} \cdot (1, 1, 2) = (3, 1, 3)$.
This means that $\mathbf{v}_4$ **is** in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$.

[8 pts] (c) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent, or linearly independent? (Justify your answer.)

Approach 1: using the linear independence equation form.
The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be linearly independent if the only solution to
$a \cdot (0, 2, 3) + b \cdot (1, 1, 2) + c(1, 0, 1) = (0, 0, 0)$
is $a = 0, b = 0, c = 0$.
Separating the vector equation into its components, we get 3 equations:

$$\begin{array}{cccll} & b & +c & = 0 & (1) \\ 2a & +b & & = 0 & (2) \\ 3a & +2b & +c & = 0 & (3) \end{array}$$

Solving through any reasonable means (substitutions or adding/subtracting rows) we find that the only solution is
$a = 0$, $b = 0$ and $c = 0$.
This means that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **linearly independent**.
Sample calculation:
Eq (1) - Eq (2) $\rightarrow -2a + c = 0 \quad (4)$.
Eq (3) - Eq (4) $\rightarrow 5a + 2b = 0$ so $b = -5a/2 \quad (5)$.
Eq (2) gives $b = -2a = 0$, which with Eq (5) gives $b = -2a = -5a/2$ and hence $a = 0$ and $b = 0$.
Eq (1) gives $c = -b = 0$.
So all $a = b = c = 0$.

Approach 2: using logic and properties of linear independence and linear combinations.
Reminder: we know from Part (a) $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
(Note that the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not trivial to prove with logic, even though the fact might seem obvious. It hinges in part on the so-far-unstated point that the subset $\{\mathbf{v}_1, \mathbf{v}_2\}$ is also linearly independent.)
Proof by contradiction:
Assume for a moment that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly **de**pendent.
Then by the relations shown in class, at least one of the vectors can be written as a linear combination of the others.

- Try $\mathbf{v}_3$ as a linear combination of the others. No: we rule that out, because we know from part (a) that $\mathbf{v}_3$ cannot be written as a linear combination of the others.
- Try $\mathbf{v}_1$ as a linear combination of the others. If that is the case, then there exist coefficients $b, c \in \mathbb{R}$ that satisfy
  $\mathbf{v}_1 = b\mathbf{v}_2 + c\mathbf{v}_3$.
  If we think about the possible coefficients though, we must have $c = 0$: if c were non-zero, then we could solve for $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, which we have already said isn't true from part (a).

  That means we must have
  $\mathbf{v}_1 = b\mathbf{v}_2$
  But now if we look at our given vectors, $\mathbf{v}_1 = (0, 2, 3)$ and $\mathbf{v}_2 = (1, 1, 2)$, they are *not* multiples of one another, which means that $\mathbf{v}_1$ cannot be a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.
- Finally, $\mathbf{v}_2$ cannot be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, by all the same logic as the previous argument about $\mathbf{v}_1$.

This means that *none* of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ can be linear combinations of the remaining vectors, which means that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be **linearly independent**.

---

2. Consider the vector space

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class.
Let $\mathbf{W}$ be a subset of $\mathbb{R}^2$ defined by $\mathbf{W} = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$.

[4 pts] (a) Does $\mathbf{W}$ contain the zero vector of $\mathbb{R}^2$? Justify your answer.

The zero element/vector in $\mathbb{R}^2$ with the usual addition and scalar multiplication operations is $(0, 0)$, and this element **is** in $\mathbf{W}$.
Proof: For the vector $(x, y) = (0, 0)$, $x^2 = 0^2 = y^2$, and so (0,0) satisfies the membership condition for $\mathbf{W}$.

[6 pts] (b) Is $\mathbf{W}$ closed under addition? If yes, prove your statement; if not, provide a counter-example.

The set $\mathbf{W}$ is **not closed** under addition.
This can be seen with any number of counter-examples, for example:

- Pick $\mathbf{u} = (1, 1); \mathbf{u} \in \mathbf{W}$ because $1^2 = 1^2$.
- Pick $\mathbf{v} = (1, -1); \mathbf{v} \in \mathbf{W}$ because $1^2 = (-1)^2$.
- However, $\mathbf{u} + \mathbf{v}$ is **not** in $\mathbf{W}$: $(1, 1) + (1, -1) = (2, 0)$, but $2^2 \neq 0^2$.

This means the sum $\mathbf{u} + \mathbf{v}$ does *not* satisfy the membership condition for $\mathbf{W}$, even though both $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{W}$.
This shows that $\mathbf{W}$ is **not closed** under addition.

[6 pts] (c) Is $\mathbf{W}$ closed under scalar multiplication? If yes, prove your statement; if not, provide a counter-example.

The set $\mathbf{W}$ **is closed** under scalar multiplication.
Proof: Consider any element $\mathbf{u} = (x, y) \in \mathbf{W}$, and any real number $a \in \mathbb{R}$.
Note: we know that $x^2 = y^2$ because $\mathbf{u} \in \mathbf{W}$.
We now compute $a \cdot \mathbf{u}$ and check to see if it is also in $\mathbf{W}$.

$$\begin{aligned} a \cdot \mathbf{u} &= a \cdot (x, y) \\ &= (ax, ay) \end{aligned}$$

This element **is** in $\mathbf{W}$, because it satisfies the membership condition (components squared equal each other):
$(ax)^2 = \underbrace{a^2x^2 = a^2y^2}_{\text{because } x^2 = y^2} = (ay)^2$.

[4 pts] (d) Determine whether or not $\mathbf{W}$ is a vector subspace of $\mathbb{R}^2$, referring to your answers in parts (a)-(c).

$\mathbf{W}$ is **not** a vector subspace of $\mathbb{R}^2$ because it failed to satisfy one of the subspace axioms.
Specifically the set $\mathbf{W}$ is not closed under addition.

---

3. Recall that $C^\infty(\mathbb{R})$ is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times. The operations on $C^\infty(\mathbb{R})$ are the usual addition and scalar multiplication of functions as seen in class. Let

$$\mathbf{W} = \left\{ f \in C^\infty(\mathbb{R}) : f''(x) = f'(x) + 2f(x) \text{ for all } x \in \mathbb{R} \right\} \subset C^\infty(\mathbb{R})$$

where f' and f'' denote the first and second derivatives of f, respectively.

[8 pts] (a) Consider the functions f_1 and f_2 in $C^\infty(\mathbb{R})$ given by

$$f_1(x) = e^{2x} + 3e^{-x}$$

and

$$f_2(x) = \cos(x)$$

for $x \in \mathbb{R}$. Determine (with justification) whether the functions f_1 and/or f_2 belong to $\mathbf{W}$.

Solution.

- For f_1, we have
$$f_1'(x) = 2e^{2x} - 3e^{-x} \text{ and } f_1''(x) = 4e^{2x} + 3e^{-x}.$$
Then,
$$\begin{aligned} f_1'(x) + 2f_1(x) &= 2e^{2x} - 3e^{-x} + 2(e^{2x} + 3e^{-x}) \\ &= 4e^{2x} + 3e^{-x} \\ &= f_1''(x), \end{aligned}$$
that is, $f_1''(x) = f_1'(x) + 2f_1(x)$ for all $x \in C^\infty(\mathbb{R})$. Therefore, we conclude that f_1 is in $\mathbf{W}$.
- For f_2, we have
$$f_2'(x) = -\sin(x) \text{ and } f_2''(x) = -\cos(x).$$
Then,
$$\begin{aligned} f_2'(x) + 2f_2(x) &= -\sin(x) + 2\cos(x) \\ &\neq -\cos(x), \end{aligned}$$
that is, $f_2''(x) = f_2'(x) + 2f_2(x)$ does **not** hold for all $x \in C^\infty(\mathbb{R})$. Therefore, we conclude that f_2 is not in $\mathbf{W}$.

[12 pts] (b) Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

Solution.

1. First, observe that the zero function $f(x) = 0$ is in $\mathbf{W}$ as

$$f''(x) = 0 \text{ and } f'(x) + 2f(x) = 0.$$

2. Let $f, g \in C^\infty(\mathbb{R})$. This implies that

$$f''(x) = f'(x) + 2f(x) \text{ and } g''(x) = g'(x) + 2g(x)$$

hold for all $x \in \mathbb{R}$. Now recall that $(f+g)(x) = f(x) + g(x)$, $(f+g)'(x) = f'(x) + g'(x)$, and $(f+g)''(x) = f''(x) + g''(x)$. Hence, we have

$$\begin{aligned}(f+g)'(x) + 2(f+g)(x) &= f'(x) + g'(x) + 2f(x) + 2g(x)\\ &= f'(x) + 2f(x) + g'(x) + 2g(x)\\ &= f''(x) + g''(x)\\ &= (f+g)''(x),\end{aligned}$$

that is, $f+g$ is in $\mathbf{W}$, i.e., $\mathbf{W}$ is closed under addition.

3. Let $\alpha \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$. Since $f \in C^\infty(\mathbb{R})$, we have $f''(x) = f'(x) + 2f(x)$. Now, recall that $(\alpha f)(x) = \alpha f(x)$, $(\alpha f)'(x) = \alpha f'(x)$, and $(\alpha f)''(x) = \alpha f''(x)$. Hence, we have

$$\begin{aligned}(\alpha f)'(x) + 2(\alpha f)(x) &= \alpha f'(x) + 2\alpha f(x)\\ &= \alpha(f'(x) + 2f(x))\\ &= \alpha f''(x)\\ &= (\alpha f)''(x),\end{aligned}$$

that is, αf is in $\mathbf{W}$, i.e., $\mathbf{W}$ is closed under scalar multiplication.

In light of $1, 2$ and 3 above, we conclude that $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

4. Answer the following questions.

[12 pts] (a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{u}$ be three vectors in a vector space $\mathbf{V}$. Show that

$$\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2} \qquad \text{if and only if} \qquad \mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2};$$

in other words, show that the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}\}$ is equal to the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if $\mathbf{u}$ belongs to the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Solution. Since the problem asks to prove an "if and only if" statement, we need to prove two things:

(i) Assuming that $\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$, we need to prove that $\mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$;

(ii) Assuming that $\mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$, we need to prove that $\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$.

Here are the proofs.

(i) Assume that $\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$. Then

$$\mathbf{u} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}, \quad \text{i.e.,} \quad \mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}\,.$$

(ii) Assume that $\mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$. To show that $\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$, we need to show two inclusions:

$$\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} \subset \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2} \quad \text{and} \quad \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2} \subset \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}}\,.$$

The second inclusion is clear (even without the assumption on $\mathbf{u}$): If $\mathbf{w} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$, then for some scalars λ_1, λ_2, we have

$$\mathbf{w} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + 0\mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}}\,,$$

proving that $\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2} \subset \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}}$.

To show the first inclusion, using the fact that $\mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$, we write $\mathbf{u} = \mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2$ for some scalars μ_1, μ_2. Let $\mathbf{w} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}}$. Then $\mathbf{w} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{u}$ for some scalars λ_1, λ_2, λ_3. Using the expression for $\mathbf{u}$ above, we obtain

$$\begin{aligned}\mathbf{w} &= \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{u} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3(\mu_1\mathbf{v}_1 + \mu_2\mathbf{v}_2)\\ &= (\lambda_1 + \lambda_3\mu_1)\mathbf{v}_1 + (\lambda_2 + \lambda_3\mu_2)\mathbf{v}_2 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2\,,\end{aligned}$$

where $\alpha_1 := \lambda_1 + \lambda_3\mu_1$ and $\alpha_2 := \lambda_2 + \lambda_3\mu_2$. Finally, the expression $\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ implies that $\mathbf{w} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$, completing the proof that $\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} \subset \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2}$. $\square$

[8 pts] (b) Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two vectors in vector space $\mathbf{V}$ such that $2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_1 + 2\mathbf{w}_2$ are linearly dependent. Show that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent as well.

Solution.

Argument 1: Since $2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_1 + 2\mathbf{w}_2$ are linearly dependent, there exist scalars λ_1 and λ_2, not both equal to zero such that

$$\lambda_1(2\mathbf{w}_1 + \mathbf{w}_2) + \lambda_2(\mathbf{w}_1 + 2\mathbf{w}_2) = \mathbf{0}.$$

Rearranging the terms above we get

$$(2\lambda_1 + \lambda_2)\mathbf{w}_1 + (\lambda_1 + 2\lambda_2)\mathbf{w}_2 = \mathbf{0}\,.$$

Setting $\mu_1 := 2\lambda_1 + \lambda_2$ and $\mu_2 := \lambda_1 + 2\lambda_2$, the equation above gives

$$\mu_1\mathbf{w}_1 + \mu_2\mathbf{w}_2 = \mathbf{0}\,.$$

Assume now that, to the contrary, $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent. Then the last equation implies that $\mu_1 = \mu_2 = 0$. In other words,

$$\begin{cases} 2\lambda_1 + \lambda_2 = 0 \\ \lambda_1 + 2\lambda_2 = 0 \end{cases}$$

Multiplying the first equation by 2 and subtracting the second one, we get $3\lambda_1 = 0$ and hence $\lambda_1 = 0$. Substituting $\lambda_1 = 0$ into the first (or the second) equation, we obtain $\lambda_2 = 0$.

To summarize, assuming that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, we arrive at the conclusion that $\lambda_1 = \lambda_2 = 0$ which contradicts their choice. This contradiction proves that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent. $\square$

Argument 2: Recall from class that two vectors are linearly dependent if and only if one of them is a scalar multiple of the other one. Hence one of $2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_1 + 2\mathbf{w}_2$ is a scalar multiple of the other one. Without loss of generality, assume that $2\mathbf{w}_1 + \mathbf{w}_2$ is a scalar multiple of $\mathbf{w}_1 + 2\mathbf{w}_2$, i.e., that

$$2\mathbf{w}_1 + \mathbf{w}_2 = \lambda(\mathbf{w}_1 + 2\mathbf{w}_2)$$

for some scalar λ. Rearranging this equation, we obtain

$$(2 - \lambda)\mathbf{w}_1 = (2\lambda - 1)\mathbf{w}_2.$$

If $\lambda \neq 2$, then $\mathbf{w}_1 = \frac{2\lambda-1}{2-\lambda}\mathbf{w}_2$ shows that $\mathbf{w}_1$ is a scalar multiple of $\mathbf{w}_2$. Hence $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent.

If $\lambda = 2$, then $\mathbf{w}_2 = \mathbf{0} = 0\mathbf{w}_1$ shows that $\mathbf{w}_2$ is a scalar multiple of $\mathbf{w}_1$ and hence $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent. (In this case we can also use the following argument: After seeing that $\mathbf{w}_2 = \mathbf{0}$, we can conclude that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent as one of them is the zero vector.) $\square$

5. Consider the set $\mathbf{V} = \{(x, y) : x, y \in \mathbb{R},\ y > 0\}$ with the following **new** addition and scalar multiplication operations, denoted by $\oplus$ and $\odot$, respectively:

Addition: For any (x_1, y_1), $(x_2, y_2) \in \mathbf{V}$,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 5,\ 3y_1y_2).$$

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $(x, y) \in \mathbf{V}$,

$$\alpha \odot (x, y) = \big(\alpha(x - 5) + 5,\ 3^{\alpha-1}y^{\alpha}\big)\,.$$

It can be proved (and you do **not** have to do this) that $\mathbf{V}$ with these operations is a vector space.

[5 pts] (a) Determine $2 \odot \big((-5, 2) \oplus (6, 1)\big)$ using the operations in $\mathbf{V}$.

Solution. Using the operations of addition and multiplications in $\mathbf{V}$ defined above, we have:

$$(-5, 2) \oplus (6, 1) = (-5 + 6 - 5, 3 \times 2 \times 1) = (-4, 6)$$

and

$$2 \odot (-4, 6) = (2(-4 - 5) + 5, 3^{2-1}\,6^2) = (2 \times (-9) + 5, 3 \times 36) = (-13, 108)\,. \quad \square$$

[5 pts] (b) Determine the zero vector $\mathbf{0}$ of $\mathbf{V}$.

Solution.
Argument 1: We have learned in class that if $\mathbf{v}$ is any vector in a vector space $(\mathbf{V}, \oplus, \odot)$, then $0 \odot \mathbf{v} = \mathbf{0}$. Thus, for any $\mathbf{v} = (x, y) \in \mathbf{V}$, we have

$$\mathbf{0} = 0 \odot (x, y) = (0(x - 5) + 5, 3^{0-1}y^0) = (5, 3^{-1}) = (5, \frac{1}{3})\,.$$

Hence $\mathbf{0} = (5, \frac{1}{3})$. $\square$

Argument 2: We can also determine $\mathbf{0}$ by the requirement (from the Axioms) that $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}$ for any vector $\mathbf{v} \in \mathbf{V}$. Letting $\mathbf{v} = (x, y) \in \mathbf{V}$ be arbitrary and denoting $\mathbf{0}$ by (a, b), we obtain the equation

$$(x, y) \oplus (a, b) = (x, y)\,.$$

Using the addition operation in $\mathbf{V}$, we obtain

$$(x, y) \oplus (a, b) = (x + a - 5, 3yb) = (x, y)\,.$$

Comparing the coordinates, we get

$$x + a - 5 = x \quad \text{and} \quad 3yb = y\,,$$

The first equation gives $a = 5$ and since $y > 0$, we can divide both sides of the second one by $3y$ to get $b = \frac{1}{3}$. Thus $\mathbf{0} = (a, b) = (5, \frac{1}{3})$. $\square$

[5 pts] (c) Given $\mathbf{v} = (x, y) \in \mathbf{V}$, determine its additive inverse; that is, find $\mathbf{w} = (z, t) \in \mathbf{V}$ such that $\mathbf{v} \oplus \mathbf{w} = \mathbf{0}$.

Solution.
Argument 1: Here we can use the fact, learned in class, that the additive inverse of any vector $\mathbf{v} \in \mathbf{V}$ is $(-1) \odot \mathbf{v}$. Thus the additive inverse of $\mathbf{v} = (x, y) \in \mathbf{V}$ is given by

$$(-1) \odot (x, y) = (-1(x-5) + 5, 3^{-1-1}y^{-1}) = (10 - x, \frac{1}{9y}).$$

Hence the additive inverse of $\mathbf{v} = (x, y)$ is $(10 - x, \frac{1}{9y})$. □

Argument 2:
Alternatively, we can find the additive inverse $\mathbf{w} = (z, t)$ of $\mathbf{v} = (x, y) \in \mathbf{V}$ from the requirement that $\mathbf{v} \oplus \mathbf{w} = \mathbf{0}$. We have

$$\mathbf{v} \oplus \mathbf{w} = (x, y) \oplus (z, t) = (x + z - 5, 3yt)$$

and since $\mathbf{0} = (5, \frac{1}{3})$, we obtain the equations

$$x + z - 5 = 5 \quad \text{and} \quad 3yt = \frac{1}{3},$$

giving us $z = 10 - x$ and $t = \frac{1}{9y}$. Hence the additive inverse of (x, y) is $(10 - x, \frac{1}{9y})$. □

[5 pts] (d) Given $\mathbf{w}_1 = (0, 3)$, $\mathbf{w}_2 = (-1, \frac{1}{3})$ and $\mathbf{w}_3 = (-5, 27)$ in $\mathbf{V}$, determine (using the operations in $\mathbf{V}$) whether or not $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Solution. We have to decide if there exist scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\lambda_1 \odot \mathbf{w}_1 \oplus \lambda_2 \odot \mathbf{w}_2 = \mathbf{w}_3.$$

We calculate

$$\lambda_1 \odot \mathbf{w}_1 \oplus \lambda_2 \odot \mathbf{w}_2 = \lambda_1 \odot (0, 3) \oplus \lambda_2 \odot (-1, \tfrac{1}{3}) = \big(\lambda_1(0-5) + 5, 3^{\lambda_1 - 1}3^{\lambda_1}\big) \oplus \big(\lambda_2(-1-5) + 5, 3^{\lambda_2 - 1}(\tfrac{1}{3})^{\lambda_2}\big)$$

$$= (5 - 5\lambda_1, 3^{2\lambda_1 - 1}) \oplus (5 - 6\lambda_2, \tfrac{1}{3}) = \big((5 - 5\lambda_1) + (5 - 6\lambda_2) - 5, 3 \times 3^{2\lambda_1 - 1} \times \tfrac{1}{3}\big) = (5 - 5\lambda_1 - 6\lambda_2, 3^{2\lambda_1 - 1}).$$

Comparing the coordinates of this vector with the coordinates of $\mathbf{w}_3 = (-5, 27)$, we arrive at the following equations for λ_1 and λ_2:

$$5 - 5\lambda_1 - 6\lambda_2 = -5 \quad \text{and} \quad 3^{2\lambda_1 - 1} = 27.$$

Since $27 = 3^3$, the second equation gives $2\lambda_1 - 1 = 3$ or $\lambda_1 = 2$. Substituting this into the first equation, we conclude that $\lambda_2 = 0$ works.

Summarizing the above, we have

$$2 \odot \mathbf{w}_1 \oplus 0 \odot \mathbf{w}_2 = \mathbf{w}_3,$$

showing that $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. □