APSC 174 – Midterm 1

February 2022

A. Ableson F. Alajaji I. Dimitrov K. Zhang

Instructions:

The exam has **five** questions, worth a total of 100 marks.

Separately write on paper your answers to each problem. At the end of the test, scan and upload your answers to each problem/question in their corresponding slot on **Crowdmark**.

To receive full credit you must show your work, clearly and in order.

Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	Total
/20	/20	/20	/20	/20	/100

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1. In the vector space

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class, consider the vectors $\mathbf{v}_1 = (0, 2, 3)$, $\mathbf{v}_2 = (1, 1, 2)$, $\mathbf{v}_3 = (1, 0, 1)$, and $\mathbf{v}_4 = (3, 1, 3)$.

[6 pts] (a) Is $\mathbf{v}_3$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? (Justify your answer.)

[6 pts] (b) Is $\mathbf{v}_4$ in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$? (Justify your answer.)

[8 pts] (c) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent, or linearly independent? (Justify your answer.)

2. Consider the vector space

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

under the usual addition and scalar multiplication operations seen in class.
Let $\mathbf{W}$ be a subset of $\mathbb{R}^2$ defined by $\mathbf{W} = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$.

[4 pts] (a) Does $\mathbf{W}$ contain the zero vector of $\mathbb{R}^2$? Justify your answer.

[6 pts] (b) Is $\mathbf{W}$ closed under addition? If yes, prove your statement; if not, provide a counter-example.

[6 pts] (c) Is $\mathbf{W}$ closed under scalar multiplication? If yes, prove your statement; if not, provide a counter-example.

[4 pts] (d) Determine whether or not $\mathbf{W}$ is a vector subspace of $\mathbb{R}^2$, referring to your answers in parts (a)-(c).

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3. Recall that $C^\infty(\mathbb{R})$ is the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times. The operations on $C^\infty(\mathbb{R})$ are the usual addition and scalar multiplication of functions as seen in class. Let

$$\mathbf{W} = \left\{ f \in C^\infty(\mathbb{R}) : f''(x) = f'(x) + 2f(x) \text{ for all } x \in \mathbb{R} \right\} \subset C^\infty(\mathbb{R})$$

where f' and f'' denote the first and second derivatives of f, respectively.

[8 pts] (a) Consider the functions f_1 and f_2 in $C^\infty(\mathbb{R})$ given by

$$f_1(x) = e^{2x} + 3e^{-x}$$

and

$$f_2(x) = \cos(x)$$

for $x \in \mathbb{R}$. Determine (with justification) whether the functions f_1 and/or f_2 belong to $\mathbf{W}$.

[12 pts] (b) Determine, with proof, whether or not $\mathbf{W}$ is a subspace of $C^\infty(\mathbb{R})$.

4. Answer the following questions.

[12 pts] (a) Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{u}$ be three vectors in a vector space $\mathbf{V}$. Show that

$$\mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2,\mathbf{u}} = \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2} \qquad \text{if and only if} \qquad \mathbf{u} \in \mathbf{S}_{\mathbf{v}_1,\mathbf{v}_2};$$

in other words, show that the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}\}$ is equal to the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if $\mathbf{u}$ belongs to the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$.

[8 pts] (b) Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two vectors in vector space $\mathbf{V}$ such that $2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_1 + 2\mathbf{w}_2$ are linearly dependent. Show that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent as well.

5. Consider the set $\mathbf{V} = \{(x, y) : x, y \in \mathbb{R},\ y > 0\}$ with the following **new** addition and scalar multiplication operations, denoted by $\oplus$ and $\odot$, respectively:

Addition: For any (x_1, y_1), $(x_2, y_2) \in \mathbf{V}$,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 5,\ 3y_1y_2).$$

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $(x, y) \in \mathbf{V}$,

$$\alpha \odot (x, y) = \left(\alpha(x - 5) + 5,\ 3^{\alpha-1}y^{\alpha}\right).$$

It can be proved (and you do **not** have to do this) that $\mathbf{V}$ with these operations is a vector space.

[5 pts] (a) Determine $2 \odot \big((-5, 2) \oplus (6, 1)\big)$ using the operations in $\mathbf{V}$.

[5 pts] (b) Determine the zero vector $\mathbf{0}$ of $\mathbf{V}$.

[5 pts] (c) Given $\mathbf{v} = (x, y) \in \mathbf{V}$, determine its additive inverse; that is, find $\mathbf{w} = (z, t) \in \mathbf{V}$ such that $\mathbf{v} \oplus \mathbf{w} = \mathbf{0}$.

[5 pts] (d) Given $\mathbf{w}_1 = (0, 3)$, $\mathbf{w}_2 = (-1, \frac{1}{3})$ and $\mathbf{w}_3 = (-5, 27)$ in $\mathbf{V}$, determine (using the operations in $\mathbf{V}$) whether or not $\mathbf{w}_3$ is a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.