STUDENT NUMBER:

APSC 174 — Midterm 2

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Instructions: The exam has six questions, worth a total of 100 marks.

Answer **all questions**, writing clearly in the space provided. If you need more room, continue to answer on the back of the **previous page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order.

No textbook, lecture notes, calculator, computer, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	**6**	Total
/15	/15	/20	/20	/15	/15	/100

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[15 pts] **1.** Let $M = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & -1 & 6 & 7 \\ 1 & 1 & 2 & 5 \end{bmatrix}$.

Use row operations to put M into row reduced echelon form (RREF). Show your steps.

Solution. The RREF of the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A possible sequence of row operations for putting the matrix into RREF is

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & -1 & 6 & 7 \\ 1 & 1 & 2 & 5 \end{bmatrix} \xrightarrow{\text{R3}-\text{R1}\mapsto\text{R3}} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & -1 & 6 & 7 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{R2}+\text{R3}\mapsto\text{R2}} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 0 & 6 & 9 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{R2}-3\text{R1}\mapsto\text{R2}}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{R2}\leftrightarrow\text{R3}} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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[15 pts] **2.** Parametrize all the solutions to the system of linear equations corresponding to the following augmented matrix. Write your parametrization in vector form (x_1, x_2, ..., x_6 are good names for the variables).

$$\left[\begin{array}{cccccc|c} 1 & 0 & 4 & 0 & 2 & 0 & -1 \\ 0 & 1 & 6 & 0 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

Solution. The matrix is already in RREF :

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 4 & 0 & 2 & 0 & -1 \\ 0 & 1 & 6 & 0 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 9 \end{array}\right] \end{array}$$

The dependent variables are x_1, x_2, x_4, and x_6, and the free variables are x_3 and x_5. Setting $x_3 = t_1$ and $x_5 = t_2$, we get the equations

$$\begin{aligned} x_1 + 4x_3 + 2x_5 &= -1 \quad \text{or} \quad & x_1 = -4x_3 - 2x_5 - 1 &= -4t_1 - 2t_2 - 1 \\ x_2 + 6x_3 + 5x_5 &= 7 \quad \text{or} \quad & x_2 = -6x_3 - 5x_5 + 7 &= -6t_1 - 5t_2 + 7 \\ x_4 + 3x_5 &= 2 \quad \text{or} \quad & x_4 = -3x_5 + 2 &= -3t_2 + 2 \\ x_6 &= 9. \end{aligned}$$

In vector form this is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -4t_1 - 2t_2 - 1 \\ -6t_1 - 5t_2 + 7 \\ t_1 \\ -3t_2 + 2 \\ t_2 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 0 \\ 2 \\ 0 \\ 9 \end{bmatrix} + t_1 \begin{bmatrix} -4 \\ -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ -5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

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3. Suppose that $L\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and we know that $L(2,3) = (3,0,2)$ and that $L(3,4) = (2,1,1)$.

[5 pts] (a) Write $(1,0)$ and $(0,1)$ as linear combinations of $(2,3)$ and $(3,4)$.

Solution. We have to find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1(2,3) + \alpha_2(3,4) = (1,0) \quad \text{and} \quad \beta_1(2,3) + \beta_2(3,4) = (0,1),$$

i.e., we have to solve the two systems of linear equations

$$\begin{array}{rcl} 2\alpha_1 + 3\alpha_2 &=& 1 \\ 3\alpha_1 + 4\alpha_2 &=& 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} 2\beta_1 + 3\beta_2 &=& 0 \\ 3\beta_1 + 4\beta_2 &=& 1. \end{array}$$

The second equation of the first system gives $\alpha_2 = -\frac{3}{4}\alpha_1$. Plugging this into the first equation we get $2\alpha_1 - \frac{9}{4}\alpha_1 = 1$, i.e., $\underline{\alpha_1 = -4}$, which then gives $\underline{\alpha_2 = 3}$.
Similarly, the first equation of the second system gives $\beta_2 = -\frac{2}{3}\beta_1$. Plugging this into the second equation gives $3\beta_1 - \frac{8}{3}\beta_1 = 1$, i.e., $\underline{\beta_1 = 3}$, which then gives $\underline{\beta_2 = -2}$. Thus

$$(1,0) = -4(2,3) + 3(3,4) \quad \text{and} \quad (0,1) = 3(2,3) - 2(3,4).$$

[10 pts] (b) Find the standard matrix for L.

Solution. The standard matrix of L has column vectors $L(1,0)$ and $L(0,1)$. From part (a) we know that $(1,0) = -4(2,3) + 3(3,4)$ and $(0,1) = 3(2,3) - 2(3,4)$. Since we also know that $L(2,3) = (3,0,2)$ and $L(3,4) = (2,1,1)$, we can use the linearity of L to compute $L(1,0)$ and $L(0,1)$ as

$$\begin{aligned} L(1,0) &= L\big(-4(2,3) + 3(3,4)\big) = -4L(2,3) + 3L(3,4) \\ &= -4(3,0,2) + 3(2,1,1) \\ &= (-6,3,-5) \end{aligned}$$

and

$$\begin{aligned} L(0,1) &= L\big(3(2,3) - 2(3,4)\big) = 3L(2,3) - 2L(3,4) \\ &= 3(3,0,2) - 2(2,1,1) \\ &= (5,-2,4). \end{aligned}$$

Thus the standard matrix for L is

$$\begin{bmatrix} -6 & 5 \\ 3 & -2 \\ -5 & 4 \end{bmatrix}.$$

[5 pts] (c) Find $L(4,5)$.

Solution. We use the linearity of L and part (b) to calculate

$$\begin{aligned} L(4,5) &= L\big(4(1,0) + 5(0,1)\big) = 4L(1,0) + 5L(0,1) \\ &= 4(-6,3,-5) + 5(5,-2,4) \\ &= (1,2,0) \end{aligned}$$

so that $\underline{L(4,5) = (1,2,0)}$.
Alternate Solution.

$$\begin{bmatrix} -6 & 5 \\ 3 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} -6 \\ 3 \\ -5 \end{bmatrix} + 5\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

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4. Let $L\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear transformation given by the rule

$$L(x, y, z, w) = (x + 2z + w,\ y + 4z + 3w,\ 2x - y - w).$$

[5 pts] (a) Find the standard matrix for L.

Solution. Since $L(1, 0, 0, 0) = (1, 0, 2)$, $L(0, 1, 0, 0) = (0, 1, -1)$, $L(0, 0, 1, 0) = (2, 4, 0)$, and $L(0, 0, 0, 1) = (1, 3, -1)$, the standard matrix A for L is

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 3 \\ 2 & -1 & 0 & -1 \end{bmatrix}.$$

[5 pts] (b) Find a basis for $\mathrm{Im}(L)$.

Solution. To find a basis for $\mathrm{Im}(L)$ we follow the procedure learned in class: first we put the standard matrix A into RREF and then identify the columns with the leading ones. The corresponding columns of A form a basis for $\mathrm{Im}(L)$. The RREF is:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 3 \\ 2 & -1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{R3}-2\text{R1}\mapsto\text{R3}} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & -1 & -4 & -3 \end{bmatrix} \xrightarrow{\text{R2}+\text{R3}\mapsto\text{R3}} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The leading ones are in the first and second columns of the RREF so the first and second columns of A are a basis for $\mathrm{Im}(L)$. That is, $\underline{((1, 0, 2), (0, 1 - 1)) \text{ is a basis for } \mathrm{Im}(L)}$.

[5 pts] (c) Find $\dim(\mathrm{Im}(L))$.

Solution. The dimension of a finite-dimensional vector space is the number of vectors in any basis for the vectors space. Since the basis for $\mathrm{Im}(L)$ we have found in part (b) contains two vectors, we get $\underline{\dim(\mathrm{Im}(L)) = 2}$.

[5 pts] (d) Find $\dim(\mathrm{Ker}(L))$.

Solution. Here we use the Rank-Nullity Theorem which states that if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\dim(\mathrm{Im}(L)) + \dim(\mathrm{Ker}(L)) = n$. Here $n = 4$ and we know from part (c) that $\dim(\mathrm{Im}(L)) = 2$, so

$$\dim(\mathrm{Ker}(L)) = 4 - \dim(\mathrm{Im}(L)) = 4 - 2 = 2,$$

i.e., $\underline{\dim(\mathrm{Ker}(L)) = 2}$.

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5. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, $L\colon \mathbf{V} \longrightarrow \mathbf{W}$ a linear transformation, and $\mathbf{v}_1,\ldots,\ \mathbf{v}_p$ vectors in $\mathbf{V}$.

[5 pts] (a) Suppose that $\alpha_1,\ldots,\ \alpha_p \in \mathbb{R}$ are such that $\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_p L(\mathbf{v}_p) = \mathbf{0}$.

Show that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \in \mathrm{Ker}(L)$.

Solution. Since L is a linear transformation, we have

$$\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_p L(\mathbf{v}_p) = L\big(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p\ \big).$$

Thus $\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_p L(\mathbf{v}_p) = \mathbf{0}$ implies $L\big(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p\big) = \mathbf{0}$, which exactly means that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \in \mathrm{Ker}(L)$.

[10 pts] (b) Suppose in addition that L is injective, and that $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_p\}$ is linearly independent.

Prove that $\{L(\mathbf{v}_1),\ L(\mathbf{v}_2),\ \ldots,\ L(\mathbf{v}_p)\}$ is linearly independent.

Solution. Let $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$ be such that

$$\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_p L(\mathbf{v}_p) = \mathbf{0}.$$

From part (a) we know that this implies $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \in \mathrm{Ker}(L)$. Also, we know from class that a linear tranformation L is injective if and only if $\mathrm{Ker}(L) = \{\mathbf{0}\}$. Thus if L is injective, we must have $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \in \{\mathbf{0}\}$, i.e.,

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p = \mathbf{0}.$$

Since $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_p\}$ is linearly independent, this gives $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$, In conclusion, we have shown that $\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_p L(\mathbf{v}_p) = \mathbf{0}$ implies $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$, which means that $\{L(\mathbf{v}_1),\ L(\mathbf{v}_2),\ \ldots,\ L(\mathbf{v}_p)\}$ is linearly independent.

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6. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomials of degree $\leqslant 2$, with real coefficients. That is, the polynomials of the form

$$a_0 + a_1 x + a_2 x^2$$

with a_0, a_1, $a_2 \in \mathbb{R}$. The operations on $\mathcal{P}_2(\mathbb{R})$ are the usual addition and scalar multiplication of polynomials.

[10 pts] (a) Find, with proof, a basis for $\mathcal{P}_2(\mathbb{R})$.

Solution. We have to find a linearly independent generating set in $\mathcal{P}_2(\mathbb{R})$. A generating set is easy to find. Consider the polynomials $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x) = x^2$, all having degree at most 2. Then the linear combination of these with coefficients $a_0, a_1, a_2 \in \mathbb{R}$ is

$$a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) = a_0 + a_1 x + a_2 x^2$$

so $\{1, x, x^2\}$ is a generating set for $\mathcal{P}_2(\mathbb{R})$.

To show that $\{1, x, x^2\}$ is linearly independent, we have to show that $a_0 + a_1 x + a_2 x^2 = 0$ for all $x \in \mathbb{R}$ is only possible when $a_0 = a_1 = a_2 = 0$. Here are three different ways to do this:

(1) Substitute $x = 0$, $x = 1$, and $x = -1$ into $a_0 + a_1 x + a_2 x^2 = 0$ to obtain the system of linear equations

$$\begin{aligned} a_1 \quad\quad\quad &= 0 \\ a_0 + a_1 + a_2 &= 0 \\ a_0 - a_1 + a_2 &= 0. \end{aligned}$$

The first row gives $a_0 = 0$ and substituting this into the second and third equations results in $a_1 + a_2 = 0$ and $-a_1 + a_2 = 0$. The unique solution of these two equations is $a_1 = 0$ and $a_2 = 0$, so the unique solution to the original system is $a_0 = a_1 = a_2 = 0$.

(2) If $a_2 \neq 0$, then $a_0 + a_1 x + a_2 x^2 = 0$ is a quadratic equation in x which has <u>at most two solutions (roots)</u> (say by the quadratic formula). If $a_2 = 0$ but $a_1 \neq 0$, then $a_0 + a_1 x + a_2 x^2 = a_0 + a_1 x$ and $a_0 + a_1 x = 0$ has <u>the unique solution $x = -a_0/a_1$</u>. Finally, if $a_1 = a_2 = 0$, but $a_0 \neq 0$, then $a_0 + a_1 x + a_2 x^2 = a_0$ so $a_0 + a_1 x + a_2 x^2 = 0$ <u>has no solution</u>. Thus $a_0 + a_1 x + a_2 x^2 = 0$ for all $x \in \mathbb{R}$ if and only if $a_0 = a_1 = a_2 = 0$.

(2) If $a_0 + a_1 x + a_2 x^2 = 0$ for all $x \in \mathbb{R}$, then all derivatives of $a_0 + a_1 x + a_2 x^2$ must also be identically zero. The first and second derivatives respectively are $a_1 + 2a_2 x$ and $2a_2$. Substituting $x = 0$ into $a_0 + a_1 x + a_2 x^2 = 0$, $a_1 + 2a_2 x = 0$ and $2a_2 = 0$ gives $a_0 = 0$, $a_1 = 0$, and $a_2 = 0$.

Thus we have shown that $\{1, x, x^2\}$ is a linearly independent generating set for $\mathcal{P}_2(\mathbb{R})$, so $(1, x, x^2)$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

[5 pts] (b) What is the dimension of $\mathcal{P}_2(\mathbb{R})$?

Solution. Since from part (a) we know that $\mathcal{P}_2(\mathbb{R})$ has a basis consisting of 3 elements, the dimension of $\mathcal{P}_2(\mathbb{R})$ is 3.