STUDENT NUMBER:

APSC 174 — Midterm 2

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Solutions

1	2	3	4	5	Total
/20	/20	/15	/20	/25	/100

STUDENT NUMBER:

1.

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 3 & 1 & 7 & 8 \\ 1 & 3 & 5 & 0 \\ 0 & 5 & 5 & -5 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF).

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 3x & + & y & + & 7z & = & 8 \\ x & + & 3y & + & 5z & = & 0 \\ & & 5y & + & 5z & = & -5 \end{array}$$

NOTE : Part (a) is relevant.

Solution Summaries for 12 Versions

Version 1

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} -2 & -1 & -5 & -5 \\ 4 & -1 & 7 & 13 \\ 3 & 2 & 8 & 7 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} (-2)x & + & (-1)y & + & (-5)z & = & -5 \\ 4x & + & (-1)y & + & 7z & = & 13 \\ 3x & + & 2y & + & 8z & = & 7 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t$$

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(— problem 1 continued —)

Version 2

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} -1 & 1 & -1 & -4 \\ 3 & 4 & 10 & 5 \\ 2 & 4 & 8 & 2 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{ccccccc} (-1)x & + & 1y & + & (-1)z & = & -4 \\ 3x & + & 4y & + & 10z & = & 5 \\ 2x & + & 4y & + & 8z & = & 2 \end{array}$$

Note : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t$$

Version 3

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 2 & 3 & 7 & 3 \\ -1 & 2 & 0 & -5 \\ 1 & 4 & 6 & -1 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{ccccccc} 2x & + & 3y & + & 7z & = & 3 \\ (-1)x & + & 2y & + & 0z & = & -5 \\ 1x & + & 4y & + & 6z & = & -1 \end{array}$$

Note : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t$$

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(— problem 1 continued —)

Version 4

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 3 & 2 & 8 & 7 \\ 2 & -1 & 3 & 7 \\ -1 & 4 & 2 & -7 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{ccccccc} 3x & + & 2y & + & 8z & = & 7 \\ 2x & + & (-1)y & + & 3z & = & 7 \\ (-1)x & + & 4y & + & 2z & = & -7 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} t$$

Version 5

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 4 & -1 & -6 & 0 \\ -1 & 1 & 3 & 3 \\ -1 & 2 & 5 & 7 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{ccccccc} 4x & + & (-1)y & + & (-6)z & = & 0 \\ (-1)x & + & 1y & + & 3z & = & 3 \\ (-1)x & + & 2y & + & 5z & = & 7 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

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(— problem 1 continued —)

Version 6

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 4 & 3 & 2 & 16 \\ 3 & 1 & -1 & 7 \\ 2 & -1 & -4 & -2 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 4x & + & 3y & + & 2z & = & 16 \\ 3x & + & 1y & + & (-1)z & = & 7 \\ 2x & + & (-1)y & + & (-4)z & = & -2 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

Version 7

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 2 & -2 & -6 & -6 \\ 3 & 3 & 3 & 15 \\ -1 & 4 & 9 & 15 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 2x & + & (-2)y & + & (-6)z & = & -6 \\ 3x & + & 3y & + & 3z & = & 15 \\ (-1)x & + & 4y & + & 9z & = & 15 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

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(— problem 1 continued —)

Version 8

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 2 & 3 & 4 & 14 \\ -2 & 3 & 8 & 10 \\ 4 & 1 & -2 & 8 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & 4z & = & 14 \\ (-2)x & + & 3y & + & 8z & = & 10 \\ 4x & + & 1y & + & (-2)z & = & 8 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

Version 9

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 1 & -2 & 3 & -7 \\ -2 & 4 & 4 & -16 \\ 1 & -2 & -1 & 5 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 1x & + & (-2)y & + & 3z & = & -7 \\ (-2)x & + & 4y & + & 4z & = & -16 \\ 1x & + & (-2)y & + & (-1)z & = & 5 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$$

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(— problem 1 continued —)

Version 10

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} -1 & 2 & -1 & 1 \\ 2 & -4 & 3 & -5 \\ 1 & -2 & 3 & -7 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} (-1)x & + & 2y & + & (-1)z & = & 1 \\ 2x & + & (-4)y & + & 3z & = & -5 \\ 1x & + & (-2)y & + & 3z & = & -7 \end{array}$$

Note : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$$

Version 11

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} -2 & 4 & 1 & -7 \\ -1 & 2 & 2 & -8 \\ -1 & 2 & -1 & 1 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} (-2)x & + & 4y & + & 1z & = & -7 \\ (-1)x & + & 2y & + & 2z & = & -8 \\ (-1)x & + & 2y & + & (-1)z & = & 1 \end{array}$$

Note : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$$

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(— problem 1 continued —)

Version 12

[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 1 & -2 & -2 & 8 \\ -1 & 2 & 4 & -14 \\ 1 & -2 & 3 & -7 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF). **You must show the row operations at each step.**

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 1x & + & (-2)y & + & (-2)z & = & 8 \\ (-1)x & + & 2y & + & 4z & = & -14 \\ 1x & + & (-2)y & + & 3z & = & -7 \end{array}$$

NOTE : Part (a) is relevant.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution vector set: possible rescalings of the following line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$$

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2. **Typical Solution** Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that $L(2,7) = (5,4)$ and $L(1,4) = (2,3)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(2,7)$ and $(1,4)$. Also write $(0,1)$ as a linear combination of $(2,7)$ and $(1,4)$.

Solution. To write $(1,0)$ as a linear combination of $(2,7)$ and $(1,4)$, we have to find scalars α_1 and α_2 such that

$$\alpha_1(2,7) + \alpha_2(1,4) = (1,0)$$

i.e., we have to solve the system of linear equations

$$\begin{aligned} 2\alpha_1 + \alpha_2 &= 1 \\ 7\alpha_1 + 4\alpha_2 &= 0. \end{aligned}$$

The second equation gives $\alpha_2 = -\frac{7}{4}\alpha_1$. Plugging this into the first equation we obtain $2\alpha_1 - \frac{7}{4}\alpha_1 = 1$, i.e., $\underline{\alpha_1 = 4}$, which then gives $\underline{\alpha_2 = -7}$. Thus

$$(1,0) = 4(2,7) - 7(1,4).$$

Similarly, to write $(0,1)$ as a linear combination of $(2,7)$ and $(1,4)$. we have to find scalars β_1 and β_2 such that

$$\beta_1(2,7) + \beta_2(1,4) = (0,1)$$

i.e., we have to solve the system of linear equations

$$\begin{aligned} 2\beta_1 + \beta_2 &= 0 \\ 7\beta_1 + 4\beta_2 &= 1. \end{aligned}$$

The first equation gives $\beta_2 = -2\beta_1$, which upon substitution into the second equation gives $7\beta_1 - 8\beta_1 = 1$, i.e., $\underline{\beta_1 = -1}$, which then gives $\underline{\beta_2 = 2}$. Thus

$$(0,1) = -(2,7) + 2(1,4).$$

[4 pts] (b) Find the standard matrix for L.

Solution. The standard matrix of L has column vectors $L(1,0)$ and $L(0,1)$. From part (a) we know that $(1,0) = 4(2,7) - 7(1,4)$ and $(0,1) = -(2,7) + 2(1,4)$. Since we also know that $L(2,7) = (5,4)$ and $L(1,4) = (2,3)$, we can use the linearity of L to compute $L(1,0)$ and $L(0,1)$ as

$$\begin{aligned} L(1,0) &= L\big(4(2,7) - 7(1,4)\big) = 4L(2,7) - 7L(1,4) \\ &= 4(5,4) - 7(2,3) \\ &= (6,-5) \end{aligned}$$

and

$$\begin{aligned} L(0,1) &= L\big(-(2,7) + 2(1,4)\big) = -L(2,7) + 2L(1,4) \\ &= -(5,4) + 2(2,3) \\ &= (-1,2). \end{aligned}$$

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Thus the standard matrix for L is

$$\begin{pmatrix} 6 & -1 \\ -5 & 2 \end{pmatrix}.$$

[5 pts] (c) Find $L(1,2)$.

Solution. We use the linearity of L and part (b) to calculate

$$\begin{aligned} L(1,2) &= L\big((1,0)+2(0,1)\big) = L(1,0)+2L(0,1) \\ &= (6,-5)+2(-1,2) \\ &= (4,-1) \end{aligned}$$

so that $\underline{L(1,2) = (4,-1)}$.

Alternate Solution.

$$\begin{pmatrix} 6 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Solution Summaries for Parts (a) and (b): 12 Versions

(Part (c) can be solved as in the Typical Solution above)

Version 1

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(2,2) = (0,8)$, and
- $L(-1,3) = (16,-12)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(2,2)$ and $(-1,3)$. Also write $(0,1)$ as a linear combination of $(2,2)$ and $(-1,3)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (0.375, -0.250) \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$(0,1) = (0.125, 0.250) \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$L = \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

$\begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 16 \\ -12 \end{bmatrix}$

Version 2

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(4,2) = (-8,20)$, and
- $L(3,2) = (-4,14)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(4,2)$ and $(3,2)$. Also write $(0,1)$ as a linear combination of $(4,2)$ and $(3,2)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (1.000, -1.000) \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$(0,1) = (-1.500, 2.000) \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$L = \begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 20 \end{bmatrix}$

$\begin{bmatrix} -4 & 4 \\ 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \end{bmatrix}$

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(— problem 2 continued —)

Version 3

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(1,4) = (12,-2)$, and
- $L(4,1) = (-12,22)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(1,4)$ and $(4,1)$. Also write $(0,1)$ as a linear combination of $(1,4)$ and $(4,1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (-0.067, 0.267) \times \begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}$

$(0,1) = (0.267, -0.067) \times \begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}$

$L = \begin{bmatrix}-4 & 4\\6 & -2\end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix}-4 & 4\\6 & -2\end{bmatrix} \cdot \begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}12\\-2\end{bmatrix}$

$\begin{bmatrix}-4 & 4\\6 & -2\end{bmatrix} \cdot \begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}-12\\22\end{bmatrix}$

Version 4

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(3,-1) = (-16,20)$, and
- $L(2,-1) = (-12,14)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(3,-1)$ and $(2,-1)$. Also write $(0,1)$ as a linear combination of $(3,-1)$ and $(2,-1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (1.000, -1.000) \times \begin{bmatrix}3\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}$

$(0,1) = (2.000, -3.000) \times \begin{bmatrix}3\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}$

$L = \begin{bmatrix}-4 & 4\\6 & -2\end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix}-4 & 4\\6 & -2\end{bmatrix} \cdot \begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}-16\\20\end{bmatrix}$

$\begin{bmatrix}-4 & 4\\6 & -2\end{bmatrix} \cdot \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}-12\\14\end{bmatrix}$

Version 5

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(2,2) = (-10,22)$, and
- $L(1,3) = (-9,21)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(2,2)$ and $(1,3)$. Also write $(0,1)$ as a linear combination of $(2,2)$ and $(1,3)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (0.750, -0.500) \times \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$

$(0,1) = (-0.250, 0.500) \times \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$

$L = \begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-10\\22\end{bmatrix}$

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-9\\21\end{bmatrix}$

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(— problem 2 continued —)

Version 6

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(-2,1) = (4,-7)$, and
- $L(2,1) = (-8,17)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(-2,1)$ and $(2,1)$. Also write $(0,1)$ as a linear combination of $(-2,1)$ and $(2,1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (-0.250, 0.250) \times \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$

$(0,1) = (0.500, 0.500) \times \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$

$L = \begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}4\\-7\end{bmatrix}$

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-8\\17\end{bmatrix}$

Version 7

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(-2,4) = (-2,8)$, and
- $L(3,2) = (-13,28)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(-2,4)$ and $(3,2)$. Also write $(0,1)$ as a linear combination of $(-2,4)$ and $(3,2)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (-0.125, 0.250) \times \begin{bmatrix}-2\\4\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}$

$(0,1) = (0.188, 0.125) \times \begin{bmatrix}-2\\4\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}$

$L = \begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}-2\\4\end{bmatrix} = \begin{bmatrix}-2\\8\end{bmatrix}$

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-13\\28\end{bmatrix}$

Version 8

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(-1,3) = (-3,9)$, and
- $L(2,-2) = (-2,2)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(-1,3)$ and $(2,-2)$. Also write $(0,1)$ as a linear combination of $(-1,3)$ and $(2,-2)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (0.500, 0.750) \times \begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}$

$(0,1) = (0.500, 0.250) \times \begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}$

$L = \begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-3\\9\end{bmatrix}$

$\begin{bmatrix}-3 & -2\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}2\\-2\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$

STUDENT NUMBER:

(— problem 2 continued —)

Version 9

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(4,2) = (12,-6)$, and
- $L(1,1) = (2,-1)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(4,2)$ and $(1,1)$. Also write $(0,1)$ as a linear combination of $(4,2)$ and $(1,1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$$(1,0) = (0.500, -1.000) \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(0,1) = (-0.500, 2.000) \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Checks: verify the following statements.

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Version 10

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(2,2) = (4,-2)$, and
- $L(3,-1) = (14,-7)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(2,2)$ and $(3,-1)$. Also write $(0,1)$ as a linear combination of $(2,2)$ and $(3,-1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$$(1,0) = (0.125, 0.250) \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$(0,1) = (0.375, -0.250) \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Checks: verify the following statements.

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

Version 11

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(4,2) = (12,-6)$, and
- $L(1,-1) = (6,-3)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(4,2)$ and $(1,-1)$. Also write $(0,1)$ as a linear combination of $(4,2)$ and $(1,-1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$$(1,0) = (0.167, 0.333) \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(0,1) = (0.167, -0.667) \times \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Checks: verify the following statements.

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

STUDENT NUMBER:

(— problem 2 continued —)

Version 12

Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation (i.e., linear map) and we know that:

- $L(-2,-1) = (-6,3)$, and
- $L(-1,1) = (-6,3)$.

[6 pts] (a) Write $(1,0)$ as a linear combination of $(-2,-1)$ and $(-1,1)$. Also write $(0,1)$ as a linear combination of $(-2,-1)$ and $(-1,1)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1,2)$.

Solution:

$(1,0) = (-0.333, -0.333) \times \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$(0,1) = (-0.333, 0.667) \times \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$L = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

Checks: verify the following statements.

$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

STUDENT NUMBER:

3. Let $\mathbf{V}$ be a vector space of dimension N; i.e., $\dim(\mathbf{V}) = N$. Determine whether or not each of the following three statements is true or false. Justify your answers.

[6 pts] (a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a generating set for $\mathbf{V}$, then $N \leqslant p$.

Solution. In both parts (a) and (b) below, we will use the **Key lemma** which states that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_a\}$ is a generating set in a vector space $\mathbf{V}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_b\}$ is a linearly independent set in $\mathbf{V}$, then $a \geq b$.

To solve part (a), we recall that since $\dim(\mathbf{V}) = N$, there is a basis $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)$ for $\mathbf{V}$. By definition of a basis, the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ is linearly independent. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a generating set, the Key Lemma (with $a = p$ and $b = N$) gives $p \geq N$, i.e., $N \leq p$.

[6 pts] (b) If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$ is a linearly independent set in $\mathbf{V}$, then $N \geqslant q$.

Since $\dim(\mathbf{V}) = N$, there is a basis $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)$ for $\mathbf{V}$. By definition of a basis, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ is a generating set for $\mathbf{V}$. Since $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$ is a linearly independent set, the Key Lemma (with $a = N$ and $b = q$) gives $N \geq q$.

[8 pts] (c) There exists a generating set of vectors for $\mathbf{V}$ with $N + 1$ elements.

Since $\dim(\mathbf{V}) = N$, there is a basis $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)$ for $\mathbf{V}$. Then, for example, the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N, \mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $\mathbf{V}$, is a generating set since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ is a generating set. This can be seen by noting that any linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ can be written as a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N, \mathbf{0}$ since for arbitrary $\alpha_1, \alpha_2, \ldots, \alpha_N, \alpha_N \mathbb{R}$,

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_N \mathbf{u}_N = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_N \mathbf{u}_N + \alpha_N \mathbf{0}$$

and thus $V = S_{(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)} = S_{(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N, \mathbf{0})}$. Also, none of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ is equal to $\mathbf{0}$ since otherwise $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ were not a linearly independent set. Thus $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N, \mathbf{0}\}$ is a generating set having $N + 1$ elements.

Note: In general if $\alpha_1, \alpha_2, \ldots, \alpha_N$ are scalars such that $(\alpha_1, \alpha_2, \ldots, \alpha_N) \neq \mathbf{e}_i$ for all $i = 1, 2, \ldots, N$, where $\mathbf{e}_i$ is the ith standard basis vector in $\mathbb{R}^N$, then $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_N \mathbf{u}_N$ is a vector that is different from each of the $\mathbf{u}_i$ and therefore $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N, \mathbf{v}\}$ is a generating set with $N + 1$ elements.

STUDENT NUMBER:

4. **Version 1** Consider the vector space $\mathbf{W}_3 = \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x > 0, y > 0, z > 0\right\}$ with the addition and scalar multiplication operations given by

Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1x_2,\, y_1y_2,\, z_1z_2)$ for any (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$.

Scalar Multiplication: $\alpha \cdot (x, y, z) = (x^\alpha,\, y^\alpha,\, z^\alpha)$ for any $\alpha \in \mathbb{R}$ and $(x, y, z) \in \mathbf{W}_3$.

Furthermore consider the function $L : \mathbf{W}_3 \to \mathbf{W}_3$ given by

$$L(x, y, z) = (xy,\, yz,\, xz) \quad \text{for } (x, y, z) \in \mathbf{W}_3.$$

[8 pts] (a) Determine, with proof, whether or not L is a linear transformation.

Solution. We need to verify whether or not L satisfies the two linearity properties of a linear map.

(i) *Addition test*: For (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$,

$$\begin{aligned} L\left((x_1, y_1, z_1) + (x_2, y_2, z_2)\right) &= L\left((x_1x_2,\, y_1y_2,\, z_1z_2)\right) \\ &= (x_1x_2y_1y_2,\, y_1y_2z_1z_2,\, x_1x_2z_1z_2) \end{aligned}$$

while

$$\begin{aligned} L\left((x_1, y_1, z_1)\right) + L\left((x_2, y_2, z_2)\right) &= (x_1y_1, y_1z_1, x_1z_1) + (x_2y_2, y_2z_2, x_2z_2) \\ &= (x_1y_1x_2y_2,\, y_1z_1y_2z_2,\, x_1z_1x_2z_2) \\ &= (x_1x_2y_1y_2,\, y_1y_2z_1z_2,\, x_1x_2z_1z_2). \end{aligned}$$

Thus

$$L\left((x_1, y_1, z_1) + (x_2, y_2, z_2)\right) = L\left((x_1, y_1, z_1)\right) + L\left((x_2, y_2, z_2)\right).$$

(ii) *Scalar multiplication test*: For $\alpha \in \mathbb{R}$ and $(x, y, z) \in \mathbf{W}_3$,

$$\begin{aligned} L\left(\alpha(x, y, z)\right) &= L\left((x^\alpha,\, y^\alpha,\, z^\alpha)\right) \\ &= (x^\alpha y^\alpha,\, y^\alpha z^\alpha,\, x^\alpha z^\alpha) \end{aligned}$$

while

$$\begin{aligned} \alpha L\left((x, y, z)\right) &= \alpha(xy,\, yz,\, xz) \\ &= ((xy)^\alpha,\, (yz)^\alpha,\, (xz)^\alpha) \\ &= (x^\alpha y^\alpha,\, y^\alpha z^\alpha,\, x^\alpha z^\alpha). \end{aligned}$$

Thus

$$L\left(\alpha(x, y, z)\right) = \alpha L\left((x, y, z)\right).$$

Since L satisfies both required linearity properties, we conclude that L is a linear transformation. □

STUDENT NUMBER:

(— problem 4 continued —)

[6 pts] (b) Specify the kernel of L, $\mathrm{Ker}(L)$, by listing its elements.

Solution. For any $\mathbf{v} = (x, y, z) \in \mathrm{Ker}(L)$, we have (by definition of the kernel) that

$$L(\mathbf{v}) = \mathbf{0},$$

where $\mathbf{0} = (1, 1, 1)$ is the zero vector of $\mathbf{W}_3$ under the stated addition and scalar multiplication operations. In other words, we have that

$$L((x, y, z)) = (1, 1, 1)$$

or equivalently, that

$$(xy,\, yz,\, xz) = (1, 1, 1).$$

Thus

$$\begin{cases} xy &= 1 \\ yz &= 1 \\ xz &= 1 \end{cases}$$

The first two equations respectively yield that $x = 1/y$ and $z = 1/y$, which upon substitution in the third equation results in $xz = 1/y^2 = 1$; i.e., $y^2 = 1$. But since $y > 0$, we must have that $y = 1$. Therefore, from the first two equations, we directly obtain that $x = z = 1$. We have hence shown that any vector $\mathbf{v} = (x, y, z) \in \mathrm{Ker}(L)$ must have the form

$$\mathbf{v} = (1, 1, 1)$$

which is the zero vector $\mathbf{0}$ of $\mathbf{W}_3$. Thus we conclude that

$$\mathrm{Ker}(L) = \{\mathbf{0}\} = \{(1, 1,)\}.$$

[3 pts] (c) Determine whether or not L is injective.

Solution. We have shown in class that a linear map $L : \mathbf{V} \to \mathbf{W}$ is injective if and only if its kernel is the trivial subspace $\mathrm{Ker}(L) = \{\mathbf{0_V}\}$, where $\mathbf{0_V}$ is the zero vector of $\mathbf{V}$.

As we have show in part (b) above that $\mathrm{Ker}(L) = \{\mathbf{0}\}$, where $\mathbf{0} = (1, 1, 1)$ is the zero vector of $\mathbf{W}_3$, we directly conclude that L is injective.

[3 pts] (d) Determine the dimension of $\mathrm{Ker}(L)$.

Solution. Since the kernel is given by the trivial subspace $\mathrm{Ker}(L) = \{\mathbf{0}\}$, i.e., it consists uniquely of the zero vector (and is hence a linearly dependent set), we directly obtain that its dimension is zero:

$$\dim(\mathrm{Ker}(L)) = 0.$$

(— problem 4 continued —)

Version 2 Consider the vector space $\mathbb{R}^3$ under the usual (component-wise) addition and scalar multiplication operations and the vector space $\mathbf{W}_3 = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x > 0, y > 0, z > 0\right\}$ with the addition and scalar multiplication operations given by

Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1x_2,\ y_1y_2,\ z_1z_2)$ for any (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$.

Scalar Multiplication: $\alpha \cdot (x, y, z) = (x^\alpha,\ y^\alpha,\ z^\alpha)$ for any $\alpha \in \mathbb{R}$ and $(x, y, z) \in \mathbf{W}_3$.

Furthermore consider the function $L : \mathbf{W}_3 \to \mathbb{R}^3$ given by

$$L(x, y, z) = (\ln(xz),\ \ln(xy),\ \ln(yz)) \quad \text{for } (x, y, z) \in \mathbf{W}_3.$$

[8 pts] (a) Determine, with proof, whether or not L is a linear transformation.

Solution. We need to verify whether or not L satisfies the two linearity properties of a linear map.

(i) *Addition test*: For (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$,

$$\begin{aligned} L\left((x_1, y_1, z_1) + (x_2, y_2, z_2)\right) &= L\left(x_1x_2,\ y_1y_2,\ z_1z_2)\right) \\ &= (\ln(x_1x_2z_1z_2),\ \ln(x_1x_2y_1y_2),\ \ln(y_1y_2z_1z_2)) \end{aligned}$$

while

$$\begin{aligned} L\left((x_1, y_1, z_1)\right) + L\left((x_2, y_2, z_2)\right) &= (\ln(x_1z_1), \ln(x_1y_1), \ln(y_1z_1)) + (\ln(x_2z_2), \ln(x_2y_2), \ln(y_2z_2)) \\ &= (\ln(x_1z_1) + \ln(x_2z_2),\ \ln(x_1y_1) + \ln(x_2y_2),\ \ln(y_1z_1) + \ln(y_2z_2)) \\ &= (\ln(x_1x_2z_1z_2),\ \ln(x_1x_2y_1y_2),\ \ln(y_1y_2z_1z_2))\,. \end{aligned}$$

Thus

$$L\left((x_1, y_1, z_1) + (x_2, y_2, z_2)\right) = L\left((x_1, y_1, z_1)\right) + L\left((x_2, y_2, z_2)\right).$$

(ii) *Scalar multiplication test*: For $\alpha \in \mathbb{R}$ and $(x, y, z) \in \mathbf{W}_3$,

$$\begin{aligned} L\left(\alpha(x, y, z)\right) &= L\left((x^\alpha,\ y^\alpha,\ z^\alpha)\right) \\ &= \left(\ln\left(x^\alpha z^\alpha\right),\ \ln\left(x^\alpha y^\alpha\right),\ \ln\left(y^\alpha z^\alpha\right)\right) \\ &= \left(\alpha \ln\left(xz\right),\ \alpha \ln\left(xy\right),\ \alpha \ln\left(yz\right)\right) \\ &= \alpha\left(\ln(xz),\ \ln(xy),\ \ln(yz)\right) \end{aligned}$$

while

$$\alpha L\left((x, y, z)\right) = \alpha\left(\ln(xz),\ \ln(xy),\ \ln(yz)\right).$$

Thus

$$L\left(\alpha(x, y, z)\right) = \alpha L\left((x, y, z)\right).$$

Since L satisfies the above linearity properties, we conclude that L is a linear transformation. $\square$

STUDENT NUMBER:

(— problem 4 continued —)

[6 pts] (b) Specify the kernel of L, $\mathrm{Ker}(L)$, by listing its elements.

Solution. For any $\mathbf{v} = (x, y, z) \in \mathrm{Ker}(L)$, we have (by definition of the kernel) that

$$L(\mathbf{v}) = \mathbf{0}',$$

where $\mathbf{0}' = (0, 0, 0)$ is the zero vector of $\mathbb{R}^3$. In other words, we have that

$$L\left((x, y, z)\right) = (0, 0, 0)$$

or equivalently, that

$$(\ln(xz), \ln(xy), \ln(yz)) = (0, 0, 0).$$

Thus $\ln(xz) = \ln(xy) = \ln(yz) = 0$, implying that

$$\begin{cases} xz &= 1 \\ xy &= 1 \\ yz &= 1 \end{cases}$$

The first two equations respectively yield that $z = 1/x$ and $y = 1/x$, which upon substitution in the third equation results in $yz = 1/x^2 = 1$; i.e., $x^2 = 1$. But since $x > 0$, we must have that $x = 1$. Therefore, from the first two equations, we directly obtain that $z = y = 1$. We have hence shown that any vector $\mathbf{v} = (x, y, z) \in \mathrm{Ker}(L)$ must have the form

$$\mathbf{v} = (1, 1, 1)$$

which is the zero vector $\mathbf{0}$ of $\mathbf{W}_3$. Thus we conclude that

$$\mathrm{Ker}(L) = \{\mathbf{0}\} = \{(1, 1, 1)\}.$$

[3 pts] (c) Determine whether or not L is injective.

Solution. We have shown in class that a linear map $L : \mathbf{V} \to \mathbf{W}$ is injective if and only if its kernel is given by the trivial subspace $\mathrm{Ker}(L) = \{\mathbf{0_V}\}$, where $\mathbf{0_V}$ is the zero vector of $\mathbf{V}$.

As we have show in part (b) above that $\mathrm{Ker}(L) = \{\mathbf{0}\}$, where $\mathbf{0} = (1, 1, 1)$ is the zero vector of $\mathbf{W}_3$, we directly conclude that L is injective.

[3 pts] (d) Determine the dimension of $\mathrm{Ker}(L)$.

Solution. Since the kernel is given by the trivial subspace $\mathrm{Ker}(L) = \{\mathbf{0}\}$, i.e., it consists uniquely of the zero vector (and is hence a linearly dependent set), we directly obtain that its dimension is zero:

$$\dim(\mathrm{Ker}(L)) = 0.$$

STUDENT NUMBER:

5. Consider the linear map $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by

$$L(f) = f'' - f,$$

where f'' denotes the second derivative of f. For instance,

$$L(x^3) = (x^3)'' - (x^3) = 6x - x^3 \text{ and } L(\sin(x)) = (\sin(x))'' - (\sin(x)) = -\sin(x) - \sin(x) = -2\sin(x).$$

[5 pts] (a) Compute $L(e^{5x})$.

Solution.

$$\begin{aligned} L(e^{ax}) &= (e^{ax})'' - e^{ax} = (ae^{ax})' - e^{ax} = a^2e^{ax} - e^{ax} \\ &= (a^2 - 1)e^{ax}. \end{aligned}$$

[5 pts] (b) Compute $L(e^{5x} + 4e^x)$.

Solution.

$$\begin{aligned} L(e^{ax} + be^x) &= (e^{ax} + be^x)'' - (e^{ax} + be^x) = (ae^{ax} + be^x)' - (e^{ax} + be^x) \\ &= (a^2e^{ax} + be^x) - (e^{ax} + be^x) \\ &= a^2e^{ax} - e^{ax} \\ &= (a^2 - 1)e^{ax}. \end{aligned}$$

[5 pts] (c) Is L injective? (Justify your answer.)

Solution. From parts (a) and (b) we see that $L(e^{ax}) = L(e^{ax} + be^x)$. If $b \neq 0$ this means that L is not injective since then there are two distinct functions e^{ax} and $e^{ax} + be^x$ in $C^\infty(\mathbb{R})$ that are mapped to the same output.

[5 pts] (d) Is $\dim(\mathrm{Ker}(L)) > 0$? (Justify your answer.)

Solution 1. We know from class that the fact$L$ is injective if and only if $\mathrm{Ker}(L) = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector in $C^\infty(\mathbb{R})$, i.e., the all zero function $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$. Since from part (c) we know that L is not injective, we have $\mathrm{Ker}(L) \neq \{\mathbf{0}\}$. But the only zero-dimensional subspace of any vector space is the trivial subspace $\{\mathbf{0}\}$, so $\mathrm{Ker}(L)$ is not zero-dimensional, i.e., $\dim(\mathrm{Ker}(L)) > 0$.

Solution 2. Using the linearity of L we see from (a) and (b) that

$$\mathbf{0} = L(e^{ax} + be^x) - L(e^{ax}) = L(be^x).$$

Thus $be^x \in \mathrm{Ker}(L)$. If $b \neq 0$, this means that $\mathrm{Ker}(L)$ contains other vectors besides $\mathbf{0}$, so $\dim(\mathrm{Ker}(L)) > 0$.