1.

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[15 pts] (a) Use row operations to put the matrix

$$\begin{pmatrix} 3 & 1 & 7 & 8 \\ 1 & 3 & 5 & 0 \\ 0 & 5 & 5 & -5 \end{pmatrix}$$

into Row Reduced Echelon Form (RREF).

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(— problem 1 continued —)

[10 pts] (b) Parameterize all the solutions to the system of linear equations below and write your answer in column vector form.

$$\begin{array}{rcrcrcr} 3x & + & y & + & 7z & = & 8 \\ x & + & 3y & + & 5z & = & 0 \\ & & 5y & + & 5z & = & -5 \end{array}$$

NOTE : Part (a) is relevant.

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2. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation (i.e., linear map) and we know that $L(2, 7) = (5, 4)$ and $L(1, 4) = (2, 3)$.

[6 pts] (a) Write $(1, 0)$ as a linear combination of $(2, 7)$ and $(1, 4)$. Also write $(0, 1)$ as a linear combination of $(2, 7)$ and $(1, 4)$.

[4 pts] (b) Find the standard matrix for L.

[5 pts] (c) Find $L(1, 2)$.

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3. Let $\mathbf{V}$ be a vector space of dimension N; i.e., $\dim(\mathbf{V}) = N$. Determine whether or not each of the following three statements is true or false. Justify your answers.

[6 pts] (a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a generating set for $\mathbf{V}$, then $N \leqslant p$.

[6 pts] (b) If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$ is a linearly independent set in $\mathbf{V}$, then $N \geqslant q$.

[8 pts] (c) There exists a generating set of vectors for $\mathbf{V}$ with $N + 1$ elements.

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4. Consider the vector space $\mathbf{W}_3 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x > 0, y > 0, z > 0\}$ with the addition and scalar multiplication operations given by

Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1x_2,\ y_1y_2,\ z_1z_2)$ for any (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbf{W}_3$.

Scalar Multiplication: $\alpha \cdot (x, y, z) = (x^\alpha,\ y^\alpha,\ z^\alpha)$ for any $\alpha \in \mathbb{R}$ and $(x, y, z) \in \mathbf{W}_3$.

Furthermore consider the function $L : \mathbf{W}_3 \to \mathbf{W}_3$ given by

$$L(x, y, z) = (xy,\ yz,\ xz) \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

[8 pts] (a) Determine, with proof, whether or not L is a linear transformation.

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(— problem 4 continued —)

[6 pts] (b) Specify the kernel of L, $\mathrm{Ker}(L)$, by listing its elements.

[3 pts] (c) Determine whether or not L is injective.

[3 pts] (d) Determine the dimension of $\mathrm{Ker}(L)$.

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5. Consider the linear map $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by

$$L(f) = f'' - f,$$

where f'' denotes the second derivative of f. For instance,

$$L(x^3) = (x^3)'' - (x^3) = 6x - x^3 \text{ and } L(\sin(x)) = (\sin(x))'' - (\sin(x)) = -\sin(x) - \sin(x) = -2\sin(x).$$

[5 pts] (a) Compute $L(e^{5x})$.

[5 pts] (b) Compute $L(e^{5x} + 4e^x)$.

[5 pts] (c) Is L injective? (Justify your answer.)

[5 pts] (d) Is $\dim(\mathrm{Ker}(L)) > 0$? (Justify your answer.)

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