APSC 174 – Midterm 2

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Solutions

Instructions:

The exam has **five** questions, worth a total of 100 marks.

Separately write on paper your answers to each problem. At the end of the test, scan and upload your answers to each problem/question in their corresponding slot on **Crowdmark**.

To receive full credit you must show your work, clearly and in order.

Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	Total
/20	/20	/20	/20	/20	/100

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1. Consider the linear system of equations shown below.

$$\begin{aligned} x + 2y + z &= 1 \\ 3x + 6y - z &= -9 \\ 2x + 4y \phantom{{}-z} &= -4 \end{aligned}$$

[4 pts] (a) Write out this system of equations using an augmented matrix.

Solution. The augmented matrix looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 6 & -1 & -9 \\ 2 & 4 & 0 & -4 \end{array}\right].$$

[12 pts] (b) Use row operations to reduce the matrix to RREF.

Solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 6 & -1 & -9 \\ 2 & 4 & 0 & -4 \end{array}\right] \xrightarrow{-3R_1 + R_2 \mapsto R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -4 & -12 \\ 2 & 4 & 0 & -4 \end{array}\right] \xrightarrow{-2R_1 + R_3 \mapsto R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & -4 & -12 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{4} R_2 \mapsto R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right] \xrightarrow{2R_2 + R_3 \mapsto R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_1 \mapsto R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

[4 pts] (c) Write out the set of solutions for the original linear system, defining and using free variables as necessary.

Solution.
Using part (a), the RREF of the matrix looks as follow:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Using the above, we can see that x, z are the leading variables while y is the free variable. We let $y = t$.

$$\begin{array}{ccccccc} x & + & 2y & & & = & -2 \\ & & & & z & = & 3 \end{array}$$

Hence, by substituting $y = t$, we get the following

$$\begin{array}{ccccc} x & = & -2 & - & 2t \\ y & = & t & & \\ z & = & 3 & & \end{array}$$

Hence, we get that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t$$

2. Suppose that $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation (map) and we know that $L(3,4) = (1,2,1)$ and $L(4,5) = (-3,1,4)$.

[5 pts] (a) Write $(1,0)$ and $(0,1)$ as linear combinations of $(3,4)$ and $(4,5)$.

Solution. In order to write $(1,0)$ as a linear combination of $(3,4)$ and $(4,5)$, we need to find α_1, β_1 such that

$$(1,0) = \alpha_1(3,4) + \beta_1(4,5).$$

This gives the equations

$$\begin{array}{rcrcr} 1 & = & 3\alpha_1 & + & 4\beta_1 \\ 0 & = & 4\alpha_1 & + & 5\beta_1. \end{array}$$

Using the second equation, we get that $\alpha_1 = \frac{-5}{4}\beta_1$. By plugging this in the first equation, we get that $1 = \frac{3.(-5)}{4}\beta_1 + 4\beta_1$, which yields $\beta_1 = 4$. Hence, $\alpha_1 = -5$. Therefore, we have

$$(1,0) = -5(3,4) + 4(4,5).$$

Similarly, in order to write $(0,1)$ as a linear combination of $(3,4)$ and $(4,5)$, we need to find α_2, β_2 such that

$$(0,1) = \alpha_2(3,4) + \beta_2(4,5).$$

This gives the equations

$$\begin{array}{rcrcr} 0 & = & 3\alpha_2 & + & 4\beta_2 \\ 1 & = & 4\alpha_2 & + & 5\beta_2. \end{array}$$

Using the first equation, we get that $\alpha_2 = \frac{-4}{3}\beta_2$. By plugging this in the second equation, we get that $1 = \frac{4.(-4)}{3}\beta_2 + 5\beta_2$, which yields $\beta_2 = -3$. Hence, $\alpha_2 = 4$. Therefore, we have

$$(0,1) = 4(3,4) - 3(4,5).$$

[5 pts] (b) Use your answer from (a) to determine $L(1,0)$ and $L(0,1)$.

Solution. Using part (a), we have $(1,0) = -5(3,4) + 4(4,5)$. Since L is a linear transformation, we have

$$\begin{aligned} L(1,0) &= L(-5(3,4) + 4(4,5)) \\ &= -5L(3,4) + 4L(4,5) \\ &= -5(1,2,1) + 4(-3,1,4) \\ &= (-17,-6,11). \end{aligned}$$

Similarity, using part (a), we have $(0,1) = 4(3,4) - 3(4,5)$. Since L is a linear transformation, we have

$$\begin{aligned} L(0,1) &= L(4(3,4) - 3(4,5)) \\ &= 4L(3,4) - 3L(4,5) \\ &= 4(1,2,1) - 3(-3,1,4) \\ &= (13,5,-8). \end{aligned}$$

[5 pts] (c) Use your answer from (b) to determine $L(-2,1)$.

Solution. Notice that $(-2,1) = -2(1,0) + (0,1)$. Using part (b), since L is a linear transformation, we have

$$\begin{aligned} L(-2,1) &= L(-2(1,0) + (0,1)) \\ &= -2L(1,0) + L(0,1) \\ &= -2(-17,-6,11) + (13,5,-8) \\ &= (47,17,-30). \end{aligned}$$

(Alternatively, we may find the standard matrix A and evaluate $A\begin{bmatrix}-2\\1\end{bmatrix}$).

[5 pts] (d) Determine the formula for $L(x,y)$ for any $(x,y) \in \mathbb{R}^2$.

Solution. Using part (b), since L is a linear transformation, we have

$$\begin{aligned} L(x,y) &= L(x(1,0) + y(0,1)) \\ &= xL(1,0) + yL(0,1) \\ &= x(-17,-6,11) + y(13,5,-8) \\ &= (-17x + 13y, -6x + 5y, 11x - 8y). \end{aligned}$$

(Alternatively, we may find the standard matrix A and evaluate $A\begin{bmatrix}x\\y\end{bmatrix}$).

3. In the vector space $C^\infty(\mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times, consider the subspace $\mathbf{H}$ *spanned* by the functions f_1 and f_2, where

$$\begin{aligned} f_1(x) &= -2 + 2x - 3x^2 \\ f_2(x) &= 1 - 3x + 2x^2 \end{aligned}$$

for $x \in \mathbb{R}$. In other words, the subspace $\mathbf{H}$ consists of all linear combinations of f_1 and f_2.

[8 pts] (a) Find a basis for $\mathbf{H}$. (Justify your answer.)

Solution 1. Since $\mathbf{H}$ is the linear span of f_1 and f_2 (i.e., any function in $\mathbf{H}$ is a linear combination of f_1 and f_2), then we directly have that $\{f_1, f_2\}$ is a *generating set* for $\mathbf{H}$. Hence if we can show that $\{f_1, f_2\}$ is a linearly independent set, then we can conclude that (f_1, f_2) is a basis for $\mathbf{H}$.

As seen in a theorem in class, two vectors are linearly dependent if and only if one of them is a scalar multiple of the other. Comparing functions f_1 with f_2 (which are both non-zero vectors), we readily see that this is *not* the case; i.e., we can verify that there is no scalar $\alpha \in \mathbb{R}$ such that $f_2 = \alpha f_1$ or $f_1 = \alpha f_2$. Thus the set $\{f_1, f_2\}$ is *not linearly dependent*; it is therefore linearly independent.

We hence conclude that (f_1, f_2) is a basis for $\mathbf{H}$.

Solution 2. Since $\mathbf{H}$ is the linear span of f_1 and f_2 (i.e., any function in $\mathbf{H}$ is a linear combination of f_1 and f_2), then we directly have that $\{f_1, f_2\}$ is a *generating set* for $\mathbf{H}$. Hence if we can show that $\{f_1, f_2\}$ is a linearly independent set, then we can conclude that (f_1, f_2) is a basis for $\mathbf{H}$.
In order to check that the set $\{f_1, f_2\}$ is linearly independent, let $\alpha, \beta \in \mathbb{R}$ be such that

$$\alpha f_1 + \beta f_2 = \mathbf{0}$$

This is equivalent to stating that for any $x \in \mathbb{R}$,

$$\alpha(-2 + 2x - 3x^2) + \beta(1 - 3x + 2x^2) = 0$$

or equivalently,

$$(-2\alpha + \beta) + (2\alpha - 3\beta)x + (-3\alpha + 2\beta)x^2 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2.$$

Since $\{1, x, x^2\}$ is a set of linearly independent vectors, we get that

$$\begin{aligned} -2\alpha + \beta &= 0 \\ 2\alpha - 3\beta &= 0 \\ -3\alpha + 2\beta &= 0 \end{aligned}$$

(Alternatively, we may get some other three equations by plugging three values for x). Solving the above system (either directly or via the RREF method), we get that

$$\alpha = \beta = 0.$$

Thus the set $\{f_1, f_2\}$ is linearly independent. We hence conclude that (f_1, f_2) is a basis for $\mathbf{H}$.

[4 pts] (b) Determine the dimension of $\mathbf{H}$. (Justify your answer.)

Solution. By part (a), we have a basis for $\mathbf{H}$ consisting of 2 elements. Hence, the dimension of $\mathbf{H}$ is $\mathbf{2}$.

[8 pts] (c) Now consider the following function f_3 in $C^\infty(\mathbb{R})$:

$$f_3(x) = -3 + x - 4x^2$$

for $x \in \mathbb{R}$. Determine (with proof) whether or not the set $\{f_1, f_2, f_3\}$ is linearly dependent.

Solution. In order to check for dependence, let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\alpha f_1 + \beta f_2 + \gamma f_3 \quad = \quad \mathbf{0}$$

This is equivalent to stating that for any $x \in \mathbb{R}$,

$$\alpha(-2 + 2x - 3x^2) + \beta(1 - 3x + 2x^2) + \gamma(-3 + x - 4x^2) \quad = \quad 0$$

or equivalently,

$$(-2\alpha + \beta - 3\gamma) + (2\alpha - 3\beta + \gamma)x + (-3\alpha + 2\beta - 4\gamma)x^2 \quad = \quad 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2.$$

Since $\{1, x, x^2\}$ is a set of linearly independent vectors, we get that

$$\begin{aligned} -2\alpha + \beta - 3\gamma &= 0 \\ 2\alpha - 3\beta + \gamma &= 0 \\ -3\alpha + 2\beta - 4\gamma &= 0 \end{aligned}$$

(Alternatively, we may get some other three equations by plugging three values for x).
The above system can be solved using the RREF method, or directly via elementary operations as follows. Adding the first two equations, we get that $\beta = -\gamma$. Plugging $\beta = -\gamma$ in the first equation gets us that $-2\alpha - 4\gamma = 0$ or that $\alpha = -2\gamma$. Now, plugging $\beta = -\gamma$ and $\alpha = -2\gamma$ in either of the three equations yields $0 = 0$. Therefore, for any choice of γ, if $\beta = -\gamma, \alpha = -2\gamma$, the three above equations will be satisfied. For example, a possible non-zero solution is $\gamma = 1$, $\beta = -1$, and $\alpha = -2$; i.e., $f_3 = 2f_1 + f_2$. Hence, the set of functions $\{f_1, f_2, f_3\}$ is *linearly dependent*.

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(x, y, z) = (x + y, y + z, x + z).$$

[4 pts] (a) Find the standard matrix of L.

Solution. The standard matrix $\mathbf{A}$ of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix with columns $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$. In our case $m = n = 3$, $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$, and

$$\begin{aligned} L(1,0,0) &= (1+0, 0+0, 1+0) = (1,0,1) \\ L(0,1,0) &= (0+1, 1+0, 0+0) = (1,1,0) \\ L(0,0,1) &= (0+0, 0+1, 0+1) = (0,1,1). \end{aligned}$$

Writing these row vectors as column vectors, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

[4 pts] (b) Determine whether the vector $(0, 1, -1)$ belongs to $\mathrm{Ker}(L)$.

Solution. By definition of the kernel, $(x, y, z) \in \mathrm{Ker}(L)$ if and only if $L(x, y, z) = (0, 0, 0) = \mathbf{0} \in \mathbb{R}^3$. For $(x, y, z) = (0, 1, -1)$,

$$L(0, 1, -1) = (0 + 1, 1 - 1, 0 - 1) = (1, 0, -1) \neq (0, 0, 0)$$

so $(0, 1, -1)$ does not belong to $\mathrm{Ker}(L)$.

[4 pts] (c) Determine whether the vector $(1, 1, 2)$ belongs to $\mathrm{Im}(L)$.

Solution. The vector $(1, 1, 2)$ belongs to $\mathrm{Im}(L)$ if and only if there exist (x, y, z) such that

$$L(x, y, z) = (x + y, y + z, x + z) = (1, 1, 2).$$

This is equivalent to asking if the system of linear equations

$$\begin{array}{ccccccc} x & + & y & & & = & 1 \\ & & y & + & z & = & 1 \\ x & & & + & z & = & 2 \end{array}$$

has a solution. This is a problem we know how to solve using the RREF method. Writing down the augmented matrix which corresponds to the system and row reducing we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We can now read off the *unique* solution $x = 1$, $y = 0$, and $z = 1$ to conclude that $(1, 1, 2)$ belongs to $\mathrm{Im}(L)$.

[4 pts] (d) Find $\mathrm{Ker}(L)$.

Solution. The kernel of L is the set of all $(x, y, z) \in \mathbb{R}^3$ such that $L(x, y, z) = (0, 0, 0)$, Equivalently, we look for all solutions to the system of linear equations

$$\begin{array}{ccccccc} x & + & y & & & = & 0 \\ & & y & + & z & = & 0 \\ x & & & + & z & = & 0 \end{array}$$

Using the RREF method as in part (c), we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which provides the *unique* solution $x = 0$, $y = 0$, and $z = 0$. Thus $\mathrm{Ker}(L) = \{(0, 0, 0)\}$, a singleton set.

[4 pts] (e) Decide, with proof, if L is injective.

By a theorem we learned in class, the linear transformation L is injective if and only if $\mathrm{Ker}(L) = \{\mathbf{0}\}$. Since in our case $\mathbf{0} = (0, 0, 0)$ and $\mathrm{Ker}(L) = \{(0, 0, 0)\}$, we obtain that L is injective.

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5. Answer the following questions.

[6 pts] (a) Consider the subspace $\mathbf{W}$ of $\mathbb{R}^3$ defined by

$$\mathbf{W} = \big\{(x,y,z) \in \mathbb{R}^3 : x - y = 0,\ 2y + z = 0\big\}.$$

Find a basis for $\mathbf{W}$. (Justify your answer.)

Solution. If $(x,y,z) \in \mathbf{W}$, then by the condition $x - y = 0$, we must have $y = x$, and by the condition $2y + z = 0$, we must have $z = -2y$, i.e., $z = -2x$. Thus

$$(x,y,z) = (x,x,-2x).$$

Conversely, for any $x \in \mathbb{R}$, we have that $(x,x,-2x) \in \mathbf{W}$, so we obtain that

$$\mathbf{W} = \{(x,x,-2x) : x \in \mathbb{R}\} = \{x(1,1,-2) : x \in \mathbb{R}\}.$$

Thus $\mathbf{W}$ is the span of the single vector $(1,1,-2) \in \mathbb{R}^3$, i.e., $\{(1,1,-2)\}$ is a generating set for $\mathbf{W}$. Since $\{(1,1,-2)\}$, as a set containing a single non-zero vector, is also linearly independent, we obtain that the single vector $(1,1,-2)$ is a basis for $\mathbf{W}$.

[8 pts] (b) Let the transformation $L : C^\infty(\mathbb{R}) \to \mathbb{R}^2$ be defined by

$$L(f) = (f(0), f(1)), \qquad f \in C^\infty(\mathbb{R}).$$

Prove or disprove that L is a linear transformation.

Solution. We check that L passes the two tests for being a linear transformation.
Addition test: Let $f, g \in C^\infty(\mathbb{R})$. Then $(f+g)(t) = f(t) + g(t)$ for all $t \in \mathbb{R}$ and therefore

$$\begin{aligned} L(f+g) &= \big(f(0)+g(0), f(1)+g(1)\big) \\ &= \big(f(0), f(1)\big) + \big(g(0), g(1)\big) \\ &= L(f) + L(g) \end{aligned}$$

so L passes the addition test.
Scalar multiplication test: Let $f \in C^\infty(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then $(\alpha f)(t) = \alpha f(t)$ for all $t \in \mathbb{R}$ and therefore

$$\begin{aligned} L(\alpha f) &= \big(\alpha f(0), \alpha f(1)\big) \\ &= \alpha\big(f(0), f(1)\big) \\ &= \alpha L(f) \end{aligned}$$

so L passes the scalar multiplication test.
Since L has passed both tests, it is a linear transformation.

[6 pts] (c) Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces and let $L : \mathbf{V} \to \mathbf{W}$ be a linear map. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in $\mathbf{V}$. Show that if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\mathrm{Ker}(L) = \{\mathbf{0_V}\}$, where $\mathbf{0_V}$ denotes the zero vector of $\mathbf{V}$, then the set $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly independent.

Solution. To prove that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly independent, let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ be such that

$$\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \alpha_3 L(\mathbf{v}_3) = \mathbf{0_W}.$$

By the linearity of L, this is equivalent to

$$L(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3) = \mathbf{0_W}$$

which gives that $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \in \mathrm{Ker}(L)$. But we know that $\mathrm{Ker}(L) = \{\mathbf{0_V}\}$, so we obtain

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0_V}.$$

Since the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, this is only possible if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. In conclusion, $\alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \alpha_3 L(\mathbf{v}_3) = \mathbf{0_W}$ implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which means that the set $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly independent.