APSC 174 – Midterm 2

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Instructions:

The exam has **five** questions, worth a total of 100 marks.

Separately write on paper your answers to each problem. At the end of the test, scan and upload your answers to each problem/question in their corresponding slot on **Crowdmark**.

To receive full credit you must show your work, clearly and in order.

Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	**2**	**3**	**4**	**5**	Total
/20	/20	/20	/20	/20	/100

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1. Consider the linear system of equations shown below.

$$\begin{aligned} x + 2y + z &= 1 \\ 3x + 6y - z &= -9 \\ 2x + 4y \qquad &= -4 \end{aligned}$$

[4 pts] (a) Write out this system of equations using an augmented matrix.

[12 pts] (b) Use row operations to reduce the matrix to RREF form.

[4 pts] (c) Write out the set of solutions for the original linear system, defining and using free variables as necessary.

2. Suppose that $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation (map) and we know that $L(3,4) = (1,2,1)$ and $L(4,5) = (-3,1,4)$.

[5 pts] (a) Write $(1,0)$ and $(0,1)$ as linear combinations of $(3,4)$ and $(4,5)$.

[5 pts] (b) Use your answer from (a) to determine $L(1,0)$ and $L(0,1)$.

[5 pts] (c) Use your answer from (b) to determine $L(-2,1)$.

[5 pts] (d) Determine the formula for $L(x,y)$ for any $(x,y) \in \mathbb{R}^2$.

3. In the vector space $C^\infty(\mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ that can be differentiated arbitrarily many times, consider the subspace $\mathbf{H}$ *spanned* by the functions f_1 and f_2, where

$$\begin{aligned} f_1(x) &= -2 + 2x - 3x^2 \\ f_2(x) &= 1 - 3x + 2x^2 \end{aligned}$$

for $x \in \mathbb{R}$. In other words, the subspace $\mathbf{H}$ consists of all linear combinations of f_1 and f_2.

[8 pts] (a) Find a basis for $\mathbf{H}$. (Justify your answer.)

[4 pts] (b) Determine the dimension of $\mathbf{H}$. (Justify your answer.)

[8 pts] (c) Now consider the following function f_3 in $C^\infty(\mathbb{R})$:

$$f_3(x) = -3 + x - 4x^2$$

for $x \in \mathbb{R}$. Determine (with proof) whether or not the set $\{f_1, f_2, f_3\}$ is linearly dependent.

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(x, y, z) = (x + y, y + z, x + z).$$

[4 pts] (a) Find the standard matrix of L.

[4 pts] (b) Determine whether the vector $(0, 1, -1)$ belongs to $\mathrm{Ker}(L)$.

[4 pts] (c) Determine whether the vector $(1, 1, 2)$ belongs to $\mathrm{Im}(L)$.

[4 pts] (d) Find $\mathrm{Ker}(L)$.

[4 pts] (e) Decide, with proof, if L is injective.

5. Answer the following questions.

[6 pts] (a) Consider the subspace $\mathbf{W}$ of $\mathbb{R}^3$ defined by

$$\mathbf{W} = \{(x, y, z) \in \mathbb{R}^3 : x - y = 0,\ 2y + z = 0\}.$$

Find a basis for $\mathbf{W}$. (Justify your answer.)

[8 pts] (b) Let the transformation $L : C^\infty(\mathbb{R}) \to \mathbb{R}^2$ be defined by

$$L(f) = (f(0), f(1)), \qquad f \in C^\infty(\mathbb{R}).$$

Prove or disprove that L is a linear transformation.

[6 pts] (c) Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces and let $L : \mathbf{V} \to \mathbf{W}$ be a linear map. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in $\mathbf{V}$. Show that if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\mathrm{Ker}(L) = \{\mathbf{0}_\mathbf{V}\}$, where $\mathbf{0}_\mathbf{V}$ denotes the zero vector of $\mathbf{V}$, then the set $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly independent.