APSC 174 – Midterm 2

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Solutions

Instructions:

The exam has **five** questions, worth a total of 100 marks.

Answer **all 5 questions**, writing clearly in the space provided, including the provided space for additional work. If you need more room, continue to answer on the **next blank page**, providing clear directions on where to find the continuation of your answer.

To receive full credit you must show your work, clearly and in order.

Correct answers without adequate explanations will not receive full marks.

No textbook, lecture notes, calculator, or other aid, is allowed.

Good luck!

1	2	3	4	5	Total
/20	/20	/20	/20	/20	/100

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1. Consider the linear system of equations shown below:

$$\begin{aligned} x + y + 2z &= 1 \\ -x + 3y + 6z &= -9 \\ 2y + 4z &= -4 \end{aligned}$$

[4 pts] (a) Write out the augmented matrix of the system above.

Solution. The augmented matrix looks as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ -1 & 3 & 6 & -9 \\ 0 & 2 & 4 & -4 \end{array}\right].$$

[12 pts] (b) Use row operations to reduce the matrix obtained in (a) to RREF.

Solution. We have the following:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ -1 & 3 & 6 & -9 \\ 0 & 2 & 4 & -4 \end{array}\right] \\ \xrightarrow{R_1 + R_2 \mapsto R_2} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 4 & 8 & -8 \\ 0 & 2 & 4 & -4 \end{array}\right] \\ \xrightarrow{\frac{1}{4}R_2 \mapsto R_2} & \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 2 & 4 & -4 \end{array}\right] \\ \xrightarrow{-R_2 + R_1 \mapsto R_1} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 2 & 4 & -4 \end{array}\right] \\ \xrightarrow{-2R_2 + R_3 \mapsto R_3} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \end{aligned}$$

[4 pts] (c) Write out the set of solutions for the original linear system, defining and using free variables as necessary.

Solution. Using part (b), the RREF of the matrix looks as follow:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Using the above, we can see that x, y are the leading variables (dependent variables) while z is the free variable (independent variable). We let $z = t$.

$$\begin{array}{ccccccc} x & & & & & = & 3 \\ & & y & + & 2z & = & -2 \end{array}$$

Hence, by substituting $z = t$, we get the following

$$\begin{array}{ccccc} x & = & 3 & & \\ y & = & -2 & - & 2t \\ z & = & & & t \end{array}$$

Hence, we get that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} t$$

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2. Suppose that $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation (map) and we know that $L(1,4) = (-2,-3,2)$ and $L(2,3) = (1,-1,4)$.

[5 pts] (a) Write $(1,0)$ and $(0,1)$ as linear combinations of $(1,4)$ and $(2,3)$.

Solution. In order to write $(1,0)$ as a linear combination of $(1,4)$ and $(2,3)$, we need to find α_1, β_1 such that

$$(1,0) = \alpha_1(1,4) + \beta_1(2,3).$$

This gives the equations

$$\begin{array}{rcrcr} 1 & = & \alpha_1 & + & 2\beta_1 \\ 0 & = & 4\alpha_1 & + & 3\beta_1. \end{array}$$

Using the second equation, we get that $\alpha_1 = \frac{-3}{4}\beta_1$. By plugging this in the first equation, we get that $1 = \frac{-3}{4}\beta_1 + 2\beta_1$, which yields $\beta_1 = \frac{4}{5}$. Hence, $\alpha_1 = -\frac{3}{5}$. Therefore, we have

$$(1,0) = -\frac{3}{5}(1,4) + \frac{4}{5}(2,3).$$

Similarly, in order to write $(0,1)$ as a linear combination of $(1,4)$ and $(2,3)$, we need to find α_2, β_2 such that

$$(0,1) = \alpha_2(1,4) + \beta_2(2,3).$$

This gives the equations

$$\begin{array}{rcrcr} 0 & = & \alpha_2 & + & 2\beta_2 \\ 1 & = & 4\alpha_2 & + & 3\beta_2. \end{array}$$

Using the first equation, we get that $\alpha_2 = -2\beta_2$. By plugging this in the second equation, we get that $1 = -8\beta_2 + 3\beta_2$, which yields $\beta_2 = -\frac{1}{5}$. Hence, $\alpha_2 = \frac{2}{5}$. Therefore, we have

$$(0,1) = \frac{2}{5}(1,4) - \frac{1}{5}(2,3).$$

[5 pts] (b) Determine $L(1,0)$ and $L(0,1)$.

Solution. Using part (a), we have $(1,0) = -\frac{3}{5}(1,4) + \frac{4}{5}(2,3)$. Since L is a linear transformation, we have

$$\begin{aligned} L(1,0) &= L\left(-\frac{3}{5}(1,4) + \frac{4}{5}(2,3)\right) \\ &= -\frac{3}{5}L(1,4) + \frac{4}{5}L(2,3) \\ &= -\frac{3}{5}(-2,-3,2) + \frac{4}{5}(1,-1,4) \\ &= (2,1,2). \end{aligned}$$

Similarity, using part (a), we have $(0,1) = \frac{2}{5}(1,4) - \frac{1}{5}(2,3)$. Since L is a linear transformation, we have

$$\begin{aligned} L(0,1) &= L\left(\frac{2}{5}(1,4) - \frac{1}{5}(2,3)\right) \\ &= \frac{2}{5}L(1,4) - \frac{1}{5}L(2,3) \\ &= \frac{2}{5}(-2,-3,2) - \frac{1}{5}(1,-1,4) \\ &= (-1,-1,0). \end{aligned}$$

[5 pts] (c) Determine $L(2,-1)$.

Solution.

Approach 1: Notice that $(2,-1) = 2(1,0) - (0,1)$. Using part (b), since L is a linear transformation, we have

$$\begin{aligned} L(2,-1) &= L(2(1,0) - (0,1)) \\ &= 2L(1,0) - L(0,1) \\ &= 2(2,1,2) - (-1,-1,0) \\ &= (5,3,4). \end{aligned}$$

Approach 2: Alternatively, we may find the standard matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$$

and evaluate

$$A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}.$$

[5 pts] (d) Determine the formula for $L(x,y)$ for any $(x,y) \in \mathbb{R}^2$.

Solution.

Approach 1: Using part (b), since L is a linear transformation, we have

$$\begin{aligned} L(x,y) &= L(x(1,0) + y(0,1)) \\ &= xL(1,0) + yL(0,1) \\ &= x(2,1,2) + y(-1,-1,0) \\ &= (2x-y, x-y, 2x). \end{aligned}$$

Approach 2: Alternatively, we may find the standard matrix A and evaluate $A\begin{bmatrix} x \\ y \end{bmatrix}$.

3. Consider the vector space $P_2(\mathbb{R})$ of polynomial functions of degree at most 2; i.e., any member $f \in P_2(\mathbb{R})$ has the form $f(x) = a + bx + cx^2$ for some real values of a, b and c.

Now consider the following polynomial functions in $P_2(\mathbb{R})$:

$$\begin{aligned} p_1(x) &= x^2 \\ p_2(x) &= x + 2x^2 \\ p_3(x) &= 1 + 2x + 3x^2 \end{aligned}$$

for $x \in \mathbb{R}$.

[7 pts] (a) Determine if the set $\{p_1, p_2, p_3\}$ is a generating set for $P_2(\mathbb{R})$.

For the set $\{p_1, p_2, p_3\}$ to be a generating set for $P_2(\mathbb{R})$, we must be able to find a linear combination of the vectors in the set that will add up to any member of $P_2(\mathbb{R})$, meaning any polynomial of the form $a + bx + cx^2$, for any real values of a, b and c. In other words, we must be able to solve

$$\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = a + bx + cx^2$$

given a, b and c. Subbing in the given functions for the p's:

$$\alpha_1(x^2) + \alpha_2(x + 2x^2) + \alpha_3(1 + 2x + 3x^2) = a + bx + cx^2.$$

Separating and matching the coefficients of each of the powers of x gives us 3 equations:

$$\begin{array}{rccccl} \text{constants:} & 0\ \alpha_1 + & 0\ \alpha_2 + & 1\ \alpha_3 & = a \\ x: & 0\ \alpha_1 + & 1\ \alpha_2 + & 2\ \alpha_3 & = b \\ x^2: & 1\ \alpha_1 + & 2\ \alpha_2 + & 3\ \alpha_3 & = c \end{array}$$

We can translate this to an augmented matrix or work with the long-form equations above to solve for the α's. Working with the augmented matrix form, we would find the RREF to be:

$$\begin{bmatrix} 1 & 0 & 0 & | & c - 2b + a \\ 0 & 1 & 0 & | & b - 2a \\ 0 & 0 & 1 & | & a \end{bmatrix}$$

which gives the specific solutions

$\alpha_1 = c - 2b + a,$
$\alpha_2 = b - 2a$
$\alpha_3 = a.$

This shows that there is always a solution for the α's for every possible a, b and c, meaning that any member of $P_2(\mathbb{R})$ can be made using a linear combination of $p_1(x)$, $p_2(x)$ and $p_3(x)$. Thus the set $\{p_1, p_2, p_3\}$ **is** a generating set for $P_2(x)$.

[7 pts] (b) Determine if the set $\{p_1, p_2, p_3\}$ is linearly independent.

To determine whether the set $\{p_1, p_2, p_3\}$ is linearly independent, we recall first that the zero vector in $P_2(\mathbb{R})$ is the all-zero constant function $f(x) = 0$ for all x, or $0 + 0x + 0x^2$.
To test for linear independence, we find solutions to the equation

$$\alpha_1(x^2) + \alpha_2(x + 2x^2) + \alpha_3(1 + 2x + 3x^2) = 0 + 0x + 0x^2.$$

But we already know from part (a) that we can solve this equation, using $a = 0, b = 0$ and $c = 0$.

Thus setting $a = b = c = 0$ in our solution in part (a), we obtain $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$ as the **only** possible solution to the above equation, meaning the set $\{p_1, p_2, p_3\}$ is linearly **independent**.

[6 pts] (c) Determine the dimension of $P_2(\mathbb{R})$.

Since the set $\{p_1, p_2, p_3\}$ is linearly independent and is a spanning set for $P_2(\mathbb{R})$, then (p_1, p_2, p_3) is a basis for $P_2(\mathbb{R})$. Since this basis has 3 vectors in it, this means that $P_2(\mathbb{R})$ is a **3-dimensional** vector space.

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4. Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation (map) defined by

$$L(x, y, z, t) = (x + 2y - t, 2x + 4y - 2t, x + 2y + z + t).$$

[4 pts] (a) Find the standard matrix of L.

Solution. The standard matrix $\mathbf{A}$ of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix with columns $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$. In our case $m = 3, n = 4$, $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, $\mathbf{e}_4 = (0, 0, 0, 1)$, and

$$\begin{aligned}
L(1,0,0,0) &= (1+2\times 0-0, 2\times 1+4\times 0-2\times 0, 1+2\times 0+0+0) = (1,2,1)\\
L(0,1,0,0) &= (0+2\times 1-0, 2\times 0+4\times 1-2\times 0, 0+2\times 1+0+0) = (2,4,2)\\
L(0,0,1,0) &= (0+2\times 0-0, 2\times 0+4\times 0-2\times 0, 0+2\times 0+1+0) = (0,0,1)\\
L(0,0,0,1) &= (0+2\times 0-1, 2\times 0+4\times 0-2\times 1, 0+2\times 0+0+1) = (-1,-2,1).
\end{aligned}$$

Writing these row vectors as column vectors, we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 0 & -2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

[2 pts] (b) Determine whether the vector $(1, 1, -2, 1)$ belongs to $\operatorname{Ker}(L)$.

Solution. By definition of the kernel, $(x, y, z, t) \in \operatorname{Ker}(L)$ if and only if $L(x, y, z, t) = (0, 0, 0) = \mathbf{0} \in \mathbb{R}^3$. For $(x, y, z, t) = (1, 1, -2, 1)$,

$$L(1,1,-2,1) = (1+2\times 1-1, 2\times 1+4\times 1-2\times 1, 1+2\times 1+(-2)+1) = (2,4,2) \neq (0,0,0)$$

so $(1, 1, -2, 1)$ does not belong to $\operatorname{Ker}(L)$.

[4 pts] (c) Determine whether the vector $(0, 0, 4)$ belongs to $\operatorname{Im}(L)$.

Solution. The vector $(0, 0, 4)$ belongs to $\operatorname{Im}(L)$ if and only if there exist (x, y, z, t) such that

$$L(x, y, z, t) = (x + 2y - t, 2x + 4y - 2t, x + 2y + z + t) = (0, 0, 4).$$

This is equivalent to asking if the system of linear equations

$$\begin{cases} x + 2y - t = 0 \\ 2x + 4y - 2t = 0 \\ x + 2y + z + t = 4 \end{cases}$$

has a solution. This is a problem we know how to solve using the RREF method. Writing down the augmented matrix which corresponds to the system and row reducing we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & -2 & 0 \\ 1 & 2 & 1 & 1 & 4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the RREF does *not* have a row of the form $(0\ 0\ 0\ 0 \,|\, 1)$, this system has solutions. Alternatively, we can show a particular solution of this system, e.g., $x = y = 0, z = 4, t = 0$. Finally, the existence of solution(s) shows that $(0, 0, 4)$ belongs to $\operatorname{Im}(L)$.

[6 pts] (d) Find two linearly independent vectors in $\mathrm{Ker}(L)$.

Solution. The kernel of L is the set of all $(x, y, z, t) \in \mathbb{R}^4$ such that $L(x, y, z, t) = (0, 0, 0)$, Equivalently, we look for all solutions to the system of linear equations

$$\begin{cases} x + 2y - t = 0 \\ 2x + 4y - 2t = 0 \\ x + 2y + z + t = 0 \end{cases}$$

Using the RREF method as in part (c), we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 2 & 4 & 0 & -2 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This shows that y and t are free (independent) variables and x and z are pivot (dependent) variables. To write down all solutions of the system, we set $y = \alpha$ and $t = \beta$ and solving for x and z we get $x = -2\alpha + \beta, z = -2\beta$. Hence the general solutions is given by

$$(x, y, z, t) = (-2\alpha + \beta, \alpha, -2\beta, \beta) = \alpha(-2, 1, 0, 0) + \beta(1, 0, -2, 1).$$

Taking $\alpha = 1, \beta = 0$ we get the vector $\mathbf{v}_1 = (-2, 1, 0, 0) \in \mathrm{Ker}(L)$ and taking $\alpha = 0, \beta = 1$ we get the vector $\mathbf{v}_2 = (1, 0, -2, 1) \in \mathrm{Ker}(L)$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because neither is a scalar multiple of the other one. Thus, $\mathbf{v}_1 = (-2, 1, 0, 0)$ and $\mathbf{v}_2 = (1, 0, -2, 1)$ are two linearly independent vectors in $\mathrm{Ker}(L)$.

[4 pts] (e) Decide, with proof, if L is injective (i.e., one-to-one).

By a theorem we learned in class, the linear transformation L is injective if and only if $\mathrm{Ker}(L) = \{\mathbf{0}\}$. Since in our case $\mathrm{Ker}(L)$ contains nonzero vectors, e.g. $\mathbf{v}_1 = (-2, 1, 0, 0)$ and $\mathbf{v}_2 = (1, 0, -2, 1)$ from part (d), we conclude that $\mathrm{Ker}(L) \neq \{\mathbf{0}\}$. Hence L is not injective.

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5. Answer the following questions.

[10 pts] (a) Recall the weird vector space seen in class $\mathbf{W}_2 = \{(x, y) : x, y \in \mathbb{R},\, x, y > 0\}$ under the following addition and scalar multiplication operations, denoted by $\oplus$ and $\odot$, respectively:

Addition: For any $(x_1, y_1), (x_2, y_2) \in \mathbf{W}_2$, we have that $(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2,\, y_1y_2)$.

Scalar Multiplication: For any $\alpha \in \mathbb{R}$, $(x, y) \in \mathbf{W}_2$, we have that $\alpha \odot (x, y) = (x^\alpha,\, y^\alpha)$.

Now consider the transformation $L : C^\infty(\mathbb{R}) \to \mathbf{W}_2$ be defined by

$$L(f) = \big(e^{f(0)}, e^{f(5)}\big), \qquad f \in C^\infty(\mathbb{R}).$$

Prove or disprove that L is a linear transformation.

Solution. We check whether L passes the two tests for being a linear transformation.
Addition test: Let $f, g \in C^\infty(\mathbb{R})$. Then $(f + g)(t) = f(t) + g(t)$ for all $t \in \mathbb{R}$ and therefore

$$\begin{aligned} L(f+g) &= \big(e^{f(0)+g(0)}, e^{f(5)+g(5)}\big) = \big(e^{f(0)}e^{g(0)}, e^{f(5)}e^{g(5)}\big) \\ &= \big(e^{f(0)}, e^{f(5)}\big) \oplus \big(e^{g(0)}, e^{g(5)}\big) = L(f) \oplus L(g), \end{aligned}$$

so L passes the addition test.
Scalar multiplication test: Let $f \in C^\infty(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then $(\alpha f)(t) = \alpha f(t)$ for all $t \in \mathbb{R}$ and therefore

$$\begin{aligned} L(\alpha f) &= \big(e^{\alpha f(0)}, e^{\alpha f(5)}\big) = \big((e^{f(0)})^\alpha, (e^{f(5)})^\alpha\big) \\ &= \alpha \odot \big(e^{f(0)}, e^{f(5)}\big) = \alpha \odot L(f), \end{aligned}$$

so L passes the scalar multiplication test.
Since L has passed both tests, it is a linear transformation.

[10 pts] (b) Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces and let $L : \mathbf{V} \to \mathbf{W}$ be a linear map. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in $\mathbf{V}$ such that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Prove or disprove that the set $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly dependent.

Solution 1. Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, there are scalars α_1, α_2 and α_3, not all of them equal to zero and such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0_V}\ .$$

Applying the Key property of linear transformations we obtain:

$$\mathbf{0_W} = L(\mathbf{0_V}) = L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \alpha_1L(\mathbf{v}_1) + \alpha_2L(\mathbf{v}_2) + \alpha_3L(\mathbf{v}_3)\ .$$

Since $\alpha_1L(\mathbf{v}_1) + \alpha_2L(\mathbf{v}_2) + \alpha_3L(\mathbf{v}_3) = \mathbf{0_W}$ and not all α_1, α_2 and α_3 equal to zero, we conclude that is $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly dependent.

Solution 2. Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent, one of them is a linear combination of the other two. Without loss of generality, we assume that

$$\mathbf{v}_3 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \ .$$

Applying the Key property of linear transformations we obtain:

$$L(\mathbf{v}_3) = L(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) \ .$$

The last equation shows that $L(\mathbf{v}_3)$ is a linear combination of $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$, proving that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly dependent.
