1. Compute these matrix multiplications:

(a) $\begin{bmatrix} 2 & -1 \\ 5 & 4 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ 3 & 5 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & -2 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & 4 \\ 7 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 1 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & 2 \\ -2 & -1 & 3 \\ 7 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 8 & 3 & 8 \\ 2 & 1 & 0 \end{bmatrix}$

**Solution.** The products are

(a) $\begin{bmatrix} 1 & -7 & -5 \\ 22 & 15 & -6 \\ 2 & -27 & -18 \end{bmatrix}$ (b) $\begin{bmatrix} -15 & 2 \\ 38 & 13 \end{bmatrix}$ (c) $\begin{bmatrix} -11 & 8 \\ 29 & -23 \end{bmatrix}$ (d) $\begin{bmatrix} 15 & 11 & 20 \\ -4 & -4 & -16 \\ 33 & 25 & 44 \end{bmatrix}$

2. Suppose we have two linear transformations $L_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $L_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by these formulas:

$$L_1(x, y, z) = (7x + 3z, 2x + y + 8z) \quad \text{and} \quad L_2(u, v) = (4u + v, 2u + 3v, -u + 5v)\,.$$

(a) Give the formulas for the composite function $L = L_2 \circ L_1$.

(b) Using these formulas, find the standard matrix $C$ for $L$.

(c) Find the standard matrix $A$ for $L_1$ and $B$ for $L_2$.

(d) Compute the matrix product $BA$ showing the details of how you computed the entries. (You should, of course, get the matrix $C$ as an answer.)

**Solutions.** We're starting with

$$T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7x + 3z \\ 2x + y + 8z \end{pmatrix}, \text{ and } T_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4u + v \\ 2u + 3v \\ -u + 5v \end{pmatrix},$$

and so

(a)

$$\begin{aligned} T_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T_2 \left( T_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \\ &= T_2 \begin{pmatrix} 7x + 3z \\ 2x + y + 8z \end{pmatrix} = \begin{pmatrix} 4(7x + 3z) + (2x + y + 8z) \\ 2(7x + 3z) + 3(2x + y + 8z) \\ -(7x + 3z) + 5(2x + y + 8z) \end{pmatrix}, \\ &= \begin{pmatrix} 30x + y + 20z \\ 20x + 3y + 30z \\ 3x + 5y + 37z \end{pmatrix}. \end{aligned}$$

(b) Plugging in the vectors $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ into the formulas, we see that

$$T_3(\vec{e}_1) = \begin{pmatrix} 30 \\ 20 \\ 3 \end{pmatrix}, \; T_3(\vec{e}_2) = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \text{ and } T_3(\vec{e}_3) = \begin{pmatrix} 20 \\ 30 \\ 37 \end{pmatrix}.$$

and so the standard matrix for $T_3$ is $C = \begin{pmatrix} 30 & 1 & 20 \\ 20 & 3 & 30 \\ 3 & 5 & 37 \end{pmatrix}$.

(c) Similarly, using the formulas for $T_1$ we get

$$T_1(\vec{e}_1) = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, \; T_1(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } T_1(\vec{e}_3) = \begin{pmatrix} 3 \\ 8 \end{pmatrix},$$

so the standard matrix for $T_1$ is $A = \begin{pmatrix} 7 & 0 & 3 \\ 2 & 1 & 8 \end{pmatrix}$, while for $T_2$ we have

$$T_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \text{ and } T_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

giving the standard matrix $B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \\ -1 & 5 \end{pmatrix}$.

(d) Multiplying, we have

$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 7 & 0 & 3 \\ 2 & 1 & 8 \end{pmatrix} = \begin{pmatrix} 30 & 1 & 20 \\ 20 & 3 & 30 \\ 3 & 5 & 37 \end{pmatrix} \qquad \text{as expected.}$$

3. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $L$ is surjective if and only if it is injective.

**Solution.** Applying Problem 1 of Tutorial 9 to the current set up, we have that the vector spaces considered in that problem are all identically equal to $\mathbb{R}^n$:

$$\mathbf{V} = \mathbf{W} = \mathbb{R}^n$$

and hence have dimension $n$. Thus we can state (from the results of that problem) that

$$L \text{ is surjective} \iff \dim(\mathrm{Im}(L)) = n$$

and

$$L \text{ is injective} \iff \dim(\mathrm{Ker}(L)) = 0.$$

But by the *Rank-Nullity Theorem*, we have that

$$\dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = n$$

implying that

$$\dim(\mathrm{Ker}(L)) = 0 \iff \dim(\mathrm{Im}(L)) = n.$$

The above equivalence together with the first two yield that:

$$L \text{ is surjective} \iff L \text{ is injective.}$$