1. Determine if the linear transformations described by the following matrices are invertible. If not, explain why, and if so, find the matrix of the inverse transformation.

(a) $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 6 \\ 0 & 3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 3 \\ 9 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 & 5 \\ 6 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 6 & 1 & 0 \\ 7 & 10 & 4 & 1 \end{bmatrix}$

**Solution.** A linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if the columns are a basis for $\mathbb{R}^m$. Since the matrix has $n$ columns, this implies that $m = n$ so the standard matrix must be a square matrix and $L$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$. Since a set of $n$ vectors in $\mathbb{R}^n$ is a basis if and only if these vectors are linearly independent, we conclude that $L$ is invertible if and only if its standard matrix is a *square matrix* and its columns are *linearly independent*.

According to the RREF method for finding a basis for the image of a linear transformation, the columns of the standard matrix $A$ of $L$ are a basis for $\mathbb{R}^n$ if and only if the columns of the RREF of $A$ are a basis for $\mathbb{R}^n$. Then the specific form of the RREF implies that $L : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (i.e., the standard matrix is invertible) if and only if the RREF of $A$ is the $n \times n$ identity matrix $I_n$.

The algoritm for finding the inverse of (the standard matrix $A$) of $L$ is to write down the $n \times 2n$ partitioned matrix $\left[\, A \,\middle|\, I_n \,\right]$ and find its RREF. It will be in the form $\left[\, I_n \,\middle|\, A^{-1} \,\right]$, where $A^{-1}$ is the inverse of $A$, i.e., the standard matrix of $L^{-1}$, the inverse of $L$.

(a) This matrix is clearly invertible: we can find its RREF and see that it is equal to $I_2$, even, thinking geometrically, see that this transformation stretches the $x$-axes by a factor of 4 and the $y$-axis by a factor of 3. Its inverse is, from either point of view, the matrix

$$\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

(b) This matrix cannot be invertible; it is not even square.

(c) This matrix is invertible; its RREF is

$$\left[\begin{array}{cc|cc} 7 & 3 & 1 & 0 \\ 9 & 4 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & -9 & 7 \end{array}\right]$$

so see that the inverse is $\begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix}$

(d) This is matrix is not invertible. Its RREF is

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

(e) This matrix isn't invertible. Its RREF is

$$\begin{bmatrix} 3 & 1 & 5 \\ 6 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{14}{3} \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix}.$$

In fact, there was no need to even compute the RREF. The original matrix has a row of zeros at the bottom, and so the RREF will also have a bottom row which is all zeros, and so it cannot be equal to $I_3$.

(f) This matrix is invertible since

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 6 & 1 & 0 & 0 & 0 & 1 & 0 \\ 7 & 10 & 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 13 & -6 & 1 & 0 \\ 0 & 0 & 0 & 1 & -29 & 14 & -4 & 1 \end{array}\right],$$

so that the inverse matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 13 & -6 & 1 & 0 \\ -29 & 14 & -4 & 1 \end{bmatrix}$.

2. Suppose that $A$ is the matrix

$$A = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 6 & 3 \end{bmatrix}.$$

(a) Find the inverse of $A$.

(b) Explain why, for any values of $a$, $b$, and $c$, the equations

$$\begin{array}{ccccccc} 5x & + & 2y & + & 4z & = & a \\ 2x & + & 3y & + & z & = & b \\ 5x & + & 6y & + & 3z & = & c \end{array}$$

always have a unique solution.

(c) Find this unique solution (in terms of $a$, $b$, and $c$).

**Solution.**

(a) We can check that $A$ is invertible, and find the inverse at the same time, by putting $A$ ito RREF:

$$\left[\begin{array}{ccc|ccc} 5 & 2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 5 & 6 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 18 & -10 \\ 0 & 1 & 0 & -1 & -5 & 3 \\ 0 & 0 & 1 & -3 & -20 & 11 \end{array}\right]$$

and so the inverse of $A$ is the matrix $A^{-1} = \begin{bmatrix} 3 & 18 & -10 \\ -1 & -5 & 3 \\ -3 & -20 & 11 \end{bmatrix}$.

(b) If $L$ is the linear map $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ corresponding to the matrix $A$, solving the system of equations is the same as finding those vectors $(x, y, z) \in \mathbb{R}^3$ with $L(x, y, z) = (a, b, c)$. Since the linear transformation $L$ is invertible (its matrix $A$ is invertible), we know that there is a unique solution $(x, y, z)$ for each $(a, b, c)$ in $\mathbb{R}^3$.

Alternatively, since the RREF of $A$ is the identity matrix $I_3$, the usual argument with the RREF shows us that there is a unique solution. This is of course really the same argument as the one above.

(c) The definition of the inverse transformation $L^{-1}$ is that it undoes what $L$ does, so that for any vector $(a, b, c)$, $L^{-1}(a, b, c)$ is exactly the vector $(x, y, z)$ such that $L(x, y, z) = (a, b, c)$. Since we already know the matrix $B$ for $L^{-1}$, we can use this to compute $(x, y, z)$ in terms of $(a, b, c)$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 3 & 18 & -10 \\ -1 & -5 & 3 \\ -3 & -20 & 11 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 18b - 10c \\ -a - 5b + 3c \\ -3a - 20b + 11c \end{pmatrix}.$$

3. Find the inverse of the following matrix:

$$A = \begin{bmatrix} 7 & 14 & -6 \\ 1 & 2 & -1 \\ 3 & 7 & -3 \end{bmatrix}$$

**Solution.**

The augmented matrix is

$$\left[\begin{array}{ccc|ccc} 7 & 14 & -6 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 3 & 7 & -3 & 0 & 0 & 1 \end{array}\right]$$

Now, arrive at the RREF form for $A$ by applying (for instance) the following operations: row 1 - 7 row 2 $\to$ row 1; row 3 - 3 row 2 $\to$ row 3; row 2 - 2 row 3 $\to$ row 2; row 1 + row 2 $\to$ row 2; swap row 1 and row 2; swap row 2 and row 3; to arrive at

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1 & -7 & 0 \end{array}\right].‘$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 1 & -7 & 0 \end{bmatrix}$$

Note: Verify by direct computation that $AA^{-1} = I_3$.

4. Find the determinant of each of the following matrices.

(a) $A = \begin{bmatrix} 7 & 4 & 6 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & -1 \\ 3 & 2 & 3 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 9 \\ 2 & 0 & 6 \end{bmatrix}$

**Solution.**

(a) $A$ is upper-triangular. For an upper triangular matrices (or lower triangular matrices, or diagonal matrices) the determinant is equal to the product of the diagonal elements of the matrix. Thus, $\det(A) = 7 \cdot 2 \cdot 7 \cdot (-6) = -588$.

(b) Applying the Laplace expansion along the first row, we obtain:

$$\begin{aligned} \det(B) &= 2\begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ &= 2(2 \cdot 3 - 2 \cdot (-1)) - 0(1 \cdot 3 - 3 \cdot (-1)) + (-1)(1 \cdot 2 - 3 \cdot 2) = 20. \end{aligned}$$

(c) We observe that the third column is a scalar multiple of the first column. Therefore, $A$ is not invertible and as a result $\det(C) = 0$.

5. Let $a \in \mathbb{R}$ be a real number and consider the following $3 \times 3$ real matrices:

$$A = \begin{bmatrix} 1 & 1 & a \\ -1 & a & 1 \\ a & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -8 \\ 3 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix},$$

(a) Compute the determinant $\det(B)$ of $B$, and use it to determine whether or not $B$ is invertible.

(b) Compute the determinant $\det(A)$ of $A$, and use it to determine all the values of $a$ for which $A$ is invertible.

(c) Let $C$ be the matrix product $C = B^2$ (recall that $B^2 = BB$). Compute $C$ and determine whether or not $C$ is invertible.

**Solution.**

(a) We can compute $\det(B)$ by doing a row expansion of the determinant along the first row, we obtain:

$$\begin{aligned}\det(B) &= 2\det\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} - 8\det\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \\ &= 2(4) - 8(1) = 0.\end{aligned}$$

Since $\det(B) = 0$ it follows that $B$ is not invertible.

(b) We can compute $\det(A)$ by doing a row expansion of the determinant along the first row, we obtain:

$$\begin{aligned}\det(A) &= 1\det\begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} -1 & 1 \\ a & 1 \end{pmatrix} + a\det\begin{pmatrix} -1 & a \\ a & 1 \end{pmatrix} \\ &= (a-1) - (-1-a) + a(-1-a^2) = a - a^3 = a(1-a^2) = a(1-a)(1+a).\end{aligned}$$

$A$ is invertible if and only $\det(A) \neq 0$ i.e. if and only if $a \neq 0, 1, -1$.

(c) Computing the matrix product $C = BB$, we obtain:

$$C = \begin{pmatrix} -4 & -8 & -24 \\ 10 & 2 & -30 \\ 6 & 3 & -9 \end{pmatrix}.$$

We have: $\det(C) = \det(BB) = \det(B)\det(B) = 0$, and hence $C$ is not invertible.