1. For each of the matrices below,

(1) Find the characteristic polynomial;

(2) Find the roots of the characteristic polynomial;

(3) For each root, find a basis for the corresponding eigenspace.

To make it easier to factor the characteristic polynomials, 5 is a root of each one.

$(a)\ \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad (b)\ \begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix} \quad (c)\ \begin{bmatrix} 3 & 2 & 1 \\ 7 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \quad (d)\ \begin{bmatrix} 6 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 9 & -4 \\ 3 & 1 \end{bmatrix}$

(a) Find the eigenvalues of A (i.e., find the characteristic polynomial and find its roots).

(b) For each of the eigenvalues from (a), find a basis for the corresponding eigenspace.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two vectors you found in (b) (in whichever order you choose), and let $\mathbf{w} = (2, 5)$.

(c) Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(d) Find a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of A.

(e) For large n, is the x-coordinate of $A^n\mathbf{w}$ positive or negative?

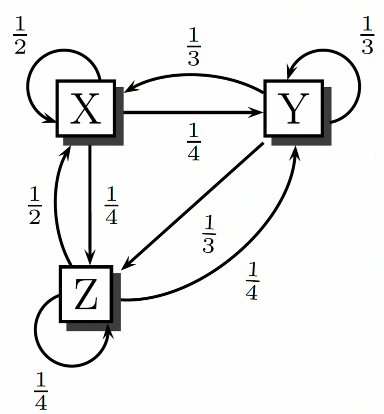
3. Consider three tanks, X, Y and Z, each holding 1000 L of water. In each minute liquid is pumped around the tanks as follows :

From X : $\frac{1}{2}$ stays in X, $\frac{1}{4}$ is pumped to Y, $\frac{1}{4}$ is pumped to Z

From Y : $\frac{1}{3}$ is pumped to X, $\frac{1}{3}$ stays in Y, $\frac{1}{3}$ is pumped to Z

From Z : $\frac{1}{2}$ is pumped to X, $\frac{1}{4}$ is pumped to Y, $\frac{1}{4}$ stays in Z

The picture below is a summary of these rules.



Suppose that at time 0, we put 90 Kg of some chemical in tank X, 50 Kg in tank Y, and 25 Kg in tank Z. (We assume the chemical dissolves completely, and does not change the volume of the tanks.)

We wish to know the amount of the chemical in each tank after n minutes have passed. Let x_n, y_n, and z_n denote the amount of the chemical in tanks X, Y, and Z respectively after n minutes. We wish to find formulas for x_n, y_n, and z_n.

We start off with $(x_0, y_0, z_0) = (90, 50, 25)$, and the rules for the procedure show that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

for all $n \geqslant 0$. Letting A be the matrix above, this means that $(x_n, y_n, z_n) = A^n(90, 50, 25)$ for all $n \geqslant 0$.

Let $\mathbf{v}_1 = (5, 3, 3)$, $\mathbf{v}_2 = (2, -1, -1)$, and $\mathbf{v}_3 = (1, 0, -1)$.

(*a*) Verify that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of A and find their eigenvalues.

(*b*) Write $(90, 50, 25)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(*c*) Find a formula for $A^n(90, 50, 25)$ in terms of the eigenvalues of A (and n of course).

(*d*) Find a formula for x_n, y_n, and z_n in terms of the eigenvalues of A.

(*e*) Find $\lim_{n\to\infty} A^n(90, 50, 25)$.

(*f*) The answer in (e) represents the amounts in tanks X, Y, and Z after a "long time". Explain, on physical grounds, why this is the answer you expect.

4. Let $A = \begin{bmatrix} \frac{1}{4} & \frac{9}{8} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$, $\mathbf{v}_1 = (3, 2)$, $\mathbf{v}_2 = (-3, 2)$, and $\mathbf{w} = (21, -2)$.

(*a*) Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A and find their eigenvalues.

(*b*) Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(*c*) Find a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of A.

(*d*) Find $\lim_{n\to\infty} A^n\mathbf{w}$.