1. For each of the matrices below,

(1) Find the characteristic polynomial;

(2) Find the roots of the characteristic polynomial;

(3) For each root, find a basis for the corresponding eigenspace.

To make it easier to factor the characteristic polynomials, 5 is a root of each one.

$(a)\ \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 3 & 2 & 1 \\ 7 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 6 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Solution.

(a) Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$.

$(a1)$ Since $A - tI_2 = \begin{bmatrix} 2-t & 4 \\ 3 & 1-t \end{bmatrix}$, the characteristic polynomial of A is

$$\det(A - tI_2) = \begin{vmatrix} 2-t & 4 \\ 3 & 1-t \end{vmatrix} = (2-t)(1-t) - (4)(3) = (t^2 - 3t + 2) - 12 = t^2 - 3t - 10.$$

$(a2)$ The characteristic polynomial factors as $(t-5)(t+2)$, so that both 5 and -2 are roots.

NOTE: To see this factorization, since we know that 5 is a root, $(t-5)$ is a factor, and we can simply divide $t^2 - 3t - 10$ by $(t-5)$ to get $(t+2)$. An even easier method is to note that if r is the unknown root, then we know that

$$t^2 - 3t - 10 = (t-5)(t-r) = t^2 - (5+r)t + 5r,$$

and comparing coefficients gives $5 + r = 3$ (so $r = -2$) or $5r = -10$ (so again $r = -2$). This method will be used in parts (b), (c), and (d) of this question without further discussion.

Finally, one could also use the quadratic formula :

$$t = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2} = \frac{3 \pm \sqrt{49}}{2} = \frac{3 \pm 7}{2} = 5,\ -2.$$

$(a3)$ We now find a basis for each eigenspace.

$\underline{\lambda = 5}$:

$$A - 5I_2 = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{bmatrix}$$

Using our algorithm a basis for $E_5 = \mathrm{Ker}(A - 5I_2)$, and therefore a basis for the eigenspace E_5 of A, is $(\frac{4}{3}, 1)$. If we multiply vectors in a basis by nonzero scalars, it remains a basis, and therefore we could also clear denominators to get $(4, 3)$ as a basis for E_5.

$\underline{\lambda = -2}$:

$$A - (-2)I_2 = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

By our algorithm, $(-1, 1)$ (or also $(1, -1)$) is a basis for $\mathrm{Ker}(A - (-2)I_2) = E_{-2}$.

(b) Let $B = \begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix}$

$(b1)$ Since $B - tI_2 = \begin{bmatrix} 8-t & 3 \\ 1 & 6-t \end{bmatrix}$ the characteristic polynomial of B is

$$\begin{vmatrix} 8-t & 3 \\ 1 & 6-t \end{vmatrix} = (8-t)(6-t) - (3)(1) = (t^2 - 14t + 48) - 3 = t^2 - 14t + 45.$$

$(b2)$ The characteristic polynomial factors as $t^2 - 14t + 45 = (t-5)(t-9)$.

$(b3)$ We now find bases for the eigenspaces.

$\underline{\lambda = 5}$:

$$B - 5I_2 = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

A basis for $E_5 = \operatorname{Ker}(B - 5I_2)$ is $(-1, 1)$.

NOTE: As a reminder, the definition of "E_5" (or "E_{-2}") depends on which matrix we are talking about. In (a), $(-1, 1)$ was a basis for the -2-eigenspace of A. Here it is also a basis for the 5-eigenspace of B.

$\underline{\lambda = 9}$:

$$B - 9I_2 = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

A basis for $E_9 = \operatorname{Ker}(B - 9I_2)$ is $(3, 1)$.

(c) Let $C = \begin{bmatrix} 3 & 2 & 1 \\ 7 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

$(c1)$ Since $C - tI_3 = \begin{bmatrix} 3-t & 2 & 1 \\ 7 & -2-t & 3 \\ 0 & 0 & 2-t \end{bmatrix}$, the characteristic polynomial of C is

$$\begin{vmatrix} 3-t & 2 & 1 \\ 7 & -2-t & 3 \\ 0 & 0 & 2-t \end{vmatrix} = (2-t) \cdot \begin{vmatrix} 3-t & 2 \\ 7 & -2-t \end{vmatrix} = (2-t) \cdot \Big((3-t)(-2-t) - (2)(7) \Big)$$

$$= (2-t)\Big((t^2 - t - 6) - 14 \Big) = (2-t)(t^2 - t - 20).$$

NOTES: Since we are eventually interested in factoring the characteristic polynomial, we should leave the factor $(2-t)$ as it is, and <u>not</u> multiply to get $(2-t)(t^2 - t - 20) = -t^3 + 3t^2 + 18t - 40$.

$(c2)$ Since $t^2 - t - 20 = (t-5)(t+4)$, the characteristic polynomial of C factors as $(2-t)(t-5)(t+4)$ with roots 2, 5, and -4.

$(c3)$ We now find bases for each of the eigenspaces.

$\underline{\lambda = 2}$:

$$C - 2I_3 = \begin{bmatrix} 1 & 2 & 1 \\ 7 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{5}{9} \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_2 = \operatorname{Ker}(C - 2I_3)$ is $(-\frac{5}{9}, -\frac{2}{9}, 1)$. Clearing denominators, $(-5, -2, 9)$ is also a basis for E_2.

$\underline{\lambda = 5}$:

$$C - 5I_3 = \begin{bmatrix} -2 & 2 & 1 \\ 7 & -7 & 3 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_5 = \mathrm{Ker}(C - 5I_3)$ is $(1, 1, 0)$.

$\underline{\lambda = -4}$:

$$C - (-4)I_3 = \begin{bmatrix} 7 & 2 & 1 \\ 7 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & \frac{2}{7} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_{-4} = \mathrm{Ker}(C - (-4)I_3)$ is $(-\frac{2}{7},\ 1,\ 0)$. Clearing denominators, $(-2, 7, 0)$ is also a basis for E_{-4}.

(d) Let $D = \begin{bmatrix} 6 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

($d1$) Since $D - tI_3 = \begin{bmatrix} 6-t & -2 & 0 \\ 2 & 1-t & 0 \\ 0 & 0 & 5-t \end{bmatrix}$, the characteristic polynomial of D is

$$\begin{vmatrix} 6-t & -2 & 0 \\ 2 & 1-t & 0 \\ 0 & 0 & 5-t \end{vmatrix} = (5-t) \begin{vmatrix} 6-t & -2 \\ 2 & 1-t \end{vmatrix} = (5-t)\Big((6-t)(1-t) - (2)(-2)\Big)$$

$$= (t-5)\Big((t^2 - 7t + 6) + 4\Big) = (t-5)(t^2 - 7t + 10).$$

($d2$) The polynomial $t^2 - 7t + 10$ factors as $(t-5)(t-2)$. (For instance, if you are stuck on factoring, there is always the quadratic formula.) Therefore the characteristic polynomial of D factors as

$$(5-t)(t^2 - 7t + 10) = (5-t)(t-5)(t-2) = -(t-5)^2(t-2),$$

and the eigenvalues of D are 5 and 2.

($d3$) We now find a basis for each eigenspace.

$\underline{\lambda = 2}$:

$$D - 2I_3 = \begin{bmatrix} 4 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_2 = \mathrm{Ker}(D - 2I_3)$ is $(\frac{1}{2},\ 1,\ 0)$. (Or, as above, $(1, 2, 0)$ is also a basis.)

$\underline{\lambda = 5}$:

$$D - 5I_3 = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for $E_5 = \mathrm{Ker}(D - 5I_3)$ is $\Big((2, 1, 0),\ (0, 0, 1)\Big)$.

2. Let $A = \begin{bmatrix} 9 & -4 \\ 3 & 1 \end{bmatrix}$

(a) Find the eigenvalues of A (i.e., find the characteristic polynomial and find its roots).

(b) For each of the eigenvalues from (a), find a basis for the corresponding eigenspace.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two vectors you found in (b) (in whichever order you choose), and let $\mathbf{w} = (2, 5)$.

(c) Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(d) Find a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of A.

(e) For large n, is the x-coordinate of $A^n\mathbf{w}$ positive or negative?

Solution.

(a) We first find the characteristic polynomial of A. Since $A - tI_2 = \begin{bmatrix} 9-t & -4 \\ 3 & 1-t \end{bmatrix}$, we compute that the characteristic polynomial of A is

$$\begin{vmatrix} 9-t & -4 \\ 3 & 1-t \end{vmatrix} = (9-t)(1-t) - (-4)(3) = (t^2 - 10t + 9) + 12 = t^2 - 10t + 21.$$

To factor this polynomial we could use the quadratic formula, which shows us that the roots are :

$$\frac{10 \pm \sqrt{(-10)^2 - 4(1)(21)}}{2} = \frac{10 \pm \sqrt{16}}{2} = \frac{10 \pm 4}{2} = 3,\ 7.$$

Alternatively, if r_1 and r_2 are the roots, then we have

$$t^2 - 10t + 21 = (t - r_1)(t - r_2) = t^2 - (r_1 + r_2)t + r_1 r_2$$

and so finding the roots amounts to solving the equations $r_1 + r_2 = 10$, $r_1 \cdot r_2 = 21$, from which one can quickly guess 3 and 7 (in either order) as the solutions.

Thus the eigenvalues of A are 3 and 7.

(b) We now find bases for each of the eigenspaces.

$\underline{\lambda = 3}$:

$$A - 3I_2 = \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}.$$

A basis for E_3 is therefore $(\frac{2}{3}, 1)$. As above, let us clear denominators and use the basis $(2, 3)$.

$\underline{\lambda = 7}$:

$$A - 7I_2 = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

A basis for E_7 is $(2, 1)$.

(c) Let $\mathbf{v}_1 = (2, 3)$ and $\mathbf{v}_2 = (2, 1)$. Writing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ means to find α_1, α_2 so that

$$\alpha_1 \cdot (2, 3) + \alpha_2 \cdot (2, 1) = (2, 5),$$

which amounts to the system of linear equations

$$\begin{array}{rcrcl} 2\alpha_1 & + & 2\alpha_2 & = & 2 \\ 3\alpha_1 & + & \alpha_2 & = & 5. \end{array}$$

We can solve this system either by fooling around with the equations, or by encoding the system in a matrix and putting the matrix into RREF. Using the second method, we get

$$\left[\begin{array}{cc|c} 2 & 2 & 2 \\ 3 & 1 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

from which we see that $\alpha_1 = 2$ and $\alpha_2 = -1$ is the unique solution; i.e., that

$$\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2.$$

(d) Since $\mathbf{v}_1$ is an eigenvector of A of eigenvalue 3, we know that $A^n\mathbf{v}_1 = 3^n\mathbf{v}_1$ for all $n \geqslant 0$. Similarly, $A^n\mathbf{v}_2 = 7^n\mathbf{v}_2$ for all $n \geqslant 0$ since $\mathbf{v}_2$ is an eigenvector of A of eigenvalue 7. Using the expression for $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ from part (c), and the linearity of A, we therefore have

$$\begin{aligned} A^n\mathbf{w} &= A^n(2\mathbf{v}_1 - \mathbf{v}_2) = 2\,(A^n\mathbf{v}_1) - (A^n\mathbf{v}_2) = 2\,(3^n\mathbf{v}_1) - (7^n\mathbf{v}_2) \\ &= 2 \cdot 3^n \cdot (2, 3) - 7^n \cdot (2, 1) = (4 \cdot 3^n - 2 \cdot 7^n,\ 6 \cdot 3^n - 7^n)\,. \end{aligned}$$

(*e*) From part (d), the x-coordinate of $A^n\mathbf{w}$ is $4 \cdot 3^n - 2 \cdot 7^n$. As n gets large, 7^n grows faster than 3^n, and so will dominate (i.e., be the largest and most important term). Therefore, for large n the x-coordinate of $A^n\mathbf{w}$ is negative.

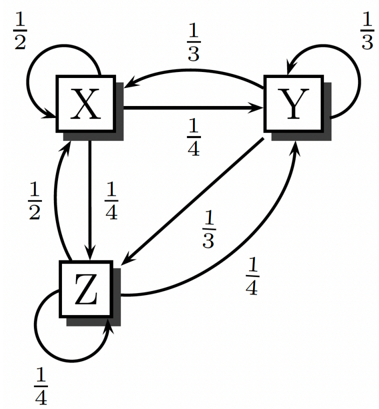
3. Consider three tanks, X, Y and Z, each holding 1000 L of water. In each minute liquid is pumped around the tanks as follows :

From X : $\frac{1}{2}$ stays in X, $\frac{1}{4}$ is pumped to Y, $\frac{1}{4}$ is pumped to Z

From Y : $\frac{1}{3}$ is pumped to X, $\frac{1}{3}$ stays in Y, $\frac{1}{3}$ is pumped to Z

From Z : $\frac{1}{2}$ is pumped to X, $\frac{1}{4}$ is pumped to Y, $\frac{1}{4}$ stays in Z

The picture below is a summary of these rules.



Suppose that at time 0, we put 90 Kg of some chemical in tank X, 50 Kg in tank Y, and 25 Kg in tank Z. (We assume the chemical dissolves completely, and does not change the volume of the tanks.)

We wish to know the amount of the chemical in each tank after n minutes have passed. Let x_n, y_n, and z_n denote the amount of the chemical in tanks X, Y, and Z respectively after n minutes. We wish to find formulas for x_n, y_n, and z_n.

We start off with $(x_0, y_0, z_0) = (90, 50, 25)$, and the rules for the procedure show that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

for all $n \geqslant 0$. Letting A be the matrix above, this means that $(x_n, y_n, z_n) = A^n(90, 50, 25)$ for all $n \geqslant 0$.

Let $\mathbf{v}_1 = (5, 3, 3)$, $\mathbf{v}_2 = (2, -1, -1)$, and $\mathbf{v}_3 = (1, 0, -1)$.

(*a*) Verify that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of A and find their eigenvalues.

(*b*) Write $(90, 50, 25)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(*c*) Find a formula for $A^n(90, 50, 25)$ in terms of the eigenvalues of A (and n of course).

(*d*) Find a formula for x_n, y_n, and z_n in terms of the eigenvalues of A.

(*e*) Find $\lim_{n \to \infty} A^n(90, 50, 25)$.

(*f*) The answer in (e) represents the amounts in tanks X, Y, and Z after a "long time". Explain, on physical grounds, why this is the answer you expect.

Solution.

(a) We have

- $$A\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} = 5 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} + 3 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + 3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix},$$
  $$= \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} = 1 \cdot \mathbf{v}_1,$$
  so $\mathbf{v}_1$ is an eigenvector of A of eigenvalue 1.
- $$A\mathbf{v}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} - 1 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} - 1 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix},$$
  $$= \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{12} \\ -\frac{1}{12} \end{pmatrix} = \frac{1}{12} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{12} \cdot \mathbf{v}_2,$$
  and so $\mathbf{v}_2$ is an eigenvector of A of eigenvalue $\frac{1}{12}$.
- $$A\mathbf{v}_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} + 0 \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} - 1 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix},$$
  $$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \cdot \mathbf{v}_3,$$
  and so $\mathbf{v}_3$ is an eigenvector of A of eigenvalue 0.

(b) The vector equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = (90, 50, 25),$$

or

$$\alpha_1(5, 3, 3) + \alpha_2(2, -1, -1) + \alpha_3(1, 0, -1) = (90, 50, 25),$$

is encoded by the augmented matrix

$$\left[\begin{array}{rrr|r} 5 & 2 & 1 & 90 \\ 3 & -1 & 0 & 50 \\ 3 & -1 & -1 & 25 \end{array}\right].$$

The RREF of this matrix is

$$\left[\begin{array}{rrr|r} 5 & 2 & 1 & 90 \\ 3 & -1 & 0 & 50 \\ 3 & -1 & -1 & 25 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 25 \end{array}\right],$$

and thus the original system of equations has the unique solution $\alpha_1 = 15$, $\alpha_2 = -5$, and $\alpha_3 = 25$. We conclude that

$$(90, 50, 25) = 15\mathbf{v}_1 - 5\mathbf{v}_2 + 25\mathbf{v}_3.$$

(c) We know that $A^n\mathbf{v}_1 = 1^n\mathbf{v}_1 = \mathbf{v}_1$, $A^n\mathbf{v}_2 = \left(\frac{1}{12}\right)^n \mathbf{v}_2$, and $A^n\mathbf{v}_3 = 0^n\mathbf{v}_3$ for all $n \geqslant 0$. Using (b) and the linearity of A^n, we therefore have

$$\begin{aligned} A^n(90, 50, 25) &= A^n(15\mathbf{v}_1 - 5\mathbf{v}_2 + 25\mathbf{v}_3) = 15(A^n\mathbf{v}_1) - 5(A^n\mathbf{v}_2) + 25(A^n\mathbf{v}_3) \\ &= 15\mathbf{v}_1 - 5(\tfrac{1}{12})^n\mathbf{v}_2 + 25 \cdot 0^n\mathbf{v}_3 \end{aligned}$$

for all $n \geqslant 0$. For $n \geqslant 1$ we can eliminate the last expression, since $0^n = 0$ for $n \neq 0$.

(*d*) For $n \geqslant 1$ we have

$$\begin{aligned}(x_n, y_n, z_n) &= A^n(90,50,25) = 15\mathbf{v}_1 - 5\left(\tfrac{1}{12}\right)^n \mathbf{v}_2 = 15(5,3,3) - 5\left(\tfrac{1}{12}\right)^n (2,-1,-1) \\ &= \left(75 - 10\left(\tfrac{1}{12}\right)^n, 45 + 5\left(\tfrac{1}{12}\right)^n, 45 + 5\left(\tfrac{1}{12}\right)^n\right).\end{aligned}$$

Comparing entries gives us the formulas

$$\begin{aligned}x_n &= 75 - 10\left(\tfrac{1}{12}\right)^n, \\ y_n &= 45 + 5\left(\tfrac{1}{12}\right)^n, \\ z_n &= 45 + 5\left(\tfrac{1}{12}\right)^n.\end{aligned}$$

(*e*) $$\begin{aligned}\lim_{n\to\infty} A^n(90,50,25) &= \lim_{n\to\infty} 15\mathbf{v}_1 - 5\left(\tfrac{1}{12}\right)^n \mathbf{v}_2 = 15\mathbf{v}_1 - 5\cdot 0\cdot \mathbf{v}_2 \\ &= 15\mathbf{v}_1 = 15(5,3,3) = (75,45,45).\end{aligned}$$

(*f*) The total amount of chemical in the system started out as 90 Kg + 50 Kg + 45 Kg = 165 Kg. Because the system is closed, none of the chemical leaves the system, and no new chemical enters, and so the total amount remains the same. This is consistent with the answer : in the long term we have a total of 75 Kg + 45 Kg + 45 Kg = 165 Kg in the tanks.

What is surprising is that the long term amounts in the tanks are different. We would expect (after a lot of mixing) that the concentration of the chemical in each tank would be the same. Since the tanks have the same volume of liquid, this should mean that the amounts should all be the same ... but they are not! How does that happen?

The answer to the puzzle is that the concentration *does* even out, but that the volume of the tanks is changing. According to the conditions of the problem "Tank X receives $\frac{1}{2}$ of tank X, $\frac{1}{3}$ of tank Y, and $\frac{1}{2}$ of tank Z". All the tanks start out with 1000 L each, and so after the first minute tank X has

$$\tfrac{1}{2}\cdot 1000 + \tfrac{1}{3}\cdot 1000 + \tfrac{1}{2}\cdot 1000 = \tfrac{4000}{3} = 1333.333333\ldots \text{ L}$$

of water. Greater than the 1000 L it started with! (Similarly the amounts in tanks Y and Z cannot stay the same — the total amount of liquid is constant.)

To see how the amounts of water in each tank change, we can use the eigenvalue method. Let $\mathbf{w} = (1000, 1000, 1000)$. Writing $\mathbf{w}$ as a linear combination $\mathbf{v}_1$, $\mathbf{v}_2$ we get

$$\mathbf{w} = \tfrac{3000}{11}\mathbf{v}_1 - \tfrac{2000}{11}\mathbf{v}_2.$$

After n minutes, the amounts of water in the three tanks is given by

$$A^n\mathbf{w} = \tfrac{3000}{11}\mathbf{v}_1 - \left(\tfrac{1}{12}\right)^n \cdot \tfrac{2000}{11}\cdot \mathbf{v}_2.$$

For large n, the limit is

$$\begin{aligned}\lim_{n\to\infty} A^n\mathbf{w} &= \tfrac{3000}{11}\mathbf{v}_1 - 0\cdot \tfrac{2000}{11}\mathbf{v}_2 = \tfrac{3000}{11}\mathbf{v}_1 \\ &= \left(\tfrac{15000}{11}, \tfrac{9000}{11}, \tfrac{9000}{11}\right) \cong (1363.636363\ldots, 818.181818\ldots, 818.181818\ldots).\end{aligned}$$

That is, the steady state of the system is to have $\cong$ 1363 L in tank X, and $\cong$ 818 L in each of tanks Y and Z.

There is 165 Kg of the chemical in 3000 L of water. If the concentration is the same in all tanks, this is amounts to $165/3000 = \frac{11}{200}$ Kg/L. With the steady-state amounts of water in each tank as above, this means we would expect to have

$$\begin{aligned}\tfrac{11}{200}\text{ Kg/L} \cdot \tfrac{15000}{11}\text{ L} &= 75 \text{ Kg in tank } X, \\ \tfrac{11}{200}\text{ Kg/L} \cdot \tfrac{9000}{11}\text{ L} &= 45 \text{ Kg in tank } Y, \text{ and} \\ \tfrac{11}{200}\text{ Kg/L} \cdot \tfrac{9000}{11}\text{ L} &= 45 \text{ Kg in tank } Z,\end{aligned}$$

which is exactly what happened.

ALTERNATE EXPLANATION : We may expect that the system will "settle down to an equilibrium" after a long period of time (or more precisely, be asymptotic to an equilibrium solution). Let $(x_\infty, y_\infty, z_\infty)$ be these equilibrium values. Since the system is not changing with time, we have $A(x_\infty, y_\infty, z_\infty) = (x_\infty, y_\infty, z_\infty)$, i.e., that $(x_\infty, y_\infty, z_\infty)$ is an eigenvector of eigenvalue 1. This implies that $(x_\infty, y_\infty, z_\infty)$ must be some multiple of $\mathbf{v}_1$, so that $(x_\infty, y_\infty, z_\infty) = \alpha\mathbf{v}_1$ for some $\alpha \in \mathbb{R}$, giving

$$(x_\infty, y_\infty, z_\infty) = \alpha\mathbf{v}_1 = \alpha(5,3,3) = (5\alpha, 3\alpha, 3\alpha).$$

As in the first argument, we also know that the total amount in the system should be 165 Kg, so $165 = 5\alpha + 3\alpha + 3\alpha = 11\alpha$, from which we conclude that $\alpha = 15$, and that $(x_\infty, y_\infty, z_\infty) = 15\mathbf{v}_1 = (75, 45, 45)$.

4. Let $A = \begin{bmatrix} \frac{1}{4} & \frac{9}{8} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$, $\mathbf{v}_1 = (3,2)$, $\mathbf{v}_2 = (-3,2)$, and $\mathbf{w} = (21,-2)$.

(a) Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A and find their eigenvalues.

(b) Write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Find a formula for $A^n\mathbf{w}$ in terms of the eigenvalues of A.

(d) Find $\lim\limits_{n\to\infty} A^n\mathbf{w}$.

Solution.

(a) We have

- $$A\mathbf{v}_1 = \begin{bmatrix} \frac{1}{4} & \frac{9}{8} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + 2\begin{pmatrix} \frac{9}{8} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1\cdot\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1\cdot\mathbf{v}_1,$$

  and so $\mathbf{v}_1$ is an eigenvector of A of eigenvalue 1.

- $$A\mathbf{v}_2 = \begin{bmatrix} \frac{1}{4} & \frac{9}{8} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = -3\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + 2\begin{pmatrix} \frac{9}{8} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -1 \end{pmatrix} = -\tfrac{1}{2}\cdot\begin{pmatrix} -3 \\ 2 \end{pmatrix} = -\tfrac{1}{2}\cdot\mathbf{v}_2,$$

  and so $\mathbf{v}_2$ is an eigenvector of A of eigenvalue $-\frac{1}{2}$.

(b) We want to find α_1 and α_2 so that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \mathbf{w},$$

i.e., so that

$$\alpha_1(3,2) + \alpha_2(-3,2) = (21,-2).$$

Converting this system of linear equations to a matrix and row reducing, we get

$$\left[\begin{array}{rr|r} 3 & -3 & 21 \\ 2 & 2 & -2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

so that $\alpha_1 = 3$ and $\alpha_2 = -4$ is the unique solution. Therefore $\mathbf{w} = 3\mathbf{v}_1 - 4\mathbf{v}_2$.

(c) $$\begin{aligned} A^n\mathbf{w} &= A^n(3\mathbf{v}_1 - 4\mathbf{v}_2) = 3(A^n\mathbf{v}_1) - 4(A^n\mathbf{v}_2) = 3\cdot 1^n\cdot\mathbf{v}_1 - 4\cdot\left(-\tfrac{1}{2}\right)^n\cdot\mathbf{v}_2 \\ &= 3\mathbf{v}_1 - 4\cdot\left(-\tfrac{1}{2}\right)^n\cdot\mathbf{v}_2. \end{aligned}$$

(d) $$\lim_{n\to\infty} A^n\mathbf{w} = \lim_{n\to\infty}\left(3\mathbf{v}_1 - 4\cdot\left(-\tfrac{1}{2}\right)^n\cdot\mathbf{v}_2\right) = 3\mathbf{v}_1 - 4\cdot 0\cdot\mathbf{v}_2 = 3\mathbf{v}_1 = 3(3,2) = (9,6).$$