Queen's University - APSC174 **Winter 2025**

Tutorial 02

1. Consider the set of all real-valued ordered n-tuples

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}$$

with $n \geq 2$. In class, we observed that $\mathbb{R}^n$ is a vector space under the following (component-wise) addition and scalar multiplication operations:

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$\alpha \cdot (x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n), \quad \alpha \in \mathbb{R}.$$

Consider the subset $\mathbf{V}$ in $\mathbb{R}^n$ given by

$$\mathbf{V} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_1 + x_2 = 0\}$$

Is $\mathbf{V}$ a vector space under the above addition and scalar multiplication operations? Support your answer by referring to the 8 axioms of a vector space.

2. The set $\mathcal{C}^\infty(\mathbb{R})$, which is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are infinitely differentiable, is a vector space under the following addition and scalar multiplication operations: for any $f_1, f_2 \in \mathcal{C}^\infty(\mathbb{R})$,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad x \in \mathbb{R},$$

for any $\alpha \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(\mathbb{R})$,

$$(\alpha \cdot f)(x) = \alpha f(x), \quad x \in \mathbb{R}.$$

(a) What is the zero vector of $\mathcal{C}^\infty(\mathbb{R})$?

(b) For any vector in $\mathcal{C}^\infty(\mathbb{R})$, determine its additive inverse.

(c) Now consider the set $\mathbf{W}$ of all functions in $\mathcal{C}^\infty(\mathbb{R})$ that satisfy $f(7) = 1$:

$$\mathbf{W} = \{f \in \mathcal{C}^\infty(\mathbb{R}) : f(7) = 1\}.$$

Is $\mathbf{W}$ a vector space under the above addition and scalar multiplication operations?

3. Consider the set of all real-valued pairs $\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ under the following **new** addition and scalar multiplication operations:

Addition: for any (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2$,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 - 1, x_2 + y_2 - 2).$$

Scalar multiplication: for any $(x_1, x_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$,

$$\alpha \cdot (x_1, x_2) = (\alpha x_1 - \alpha + 1, \alpha x_2 - 2\alpha + 2).$$

Show that $\mathbb{R}^2$ is a vector space under the above operations by demonstrating that each of the 8 axioms of a vector space is satisfied.

4. Consider the vector space

$$\mathbf{U} = \{(x, y, z) : x, y, z \in \mathbb{R}, x > 0, y > 0, z > 0\}$$

under the following addition and scalar multiplication operations:

Addition: for any (x_1, x_2, x_3) and (y_1, y_2, y_3) in $\mathbf{U}$,

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 y_1, x_2 y_2, x_3 y_3).$$

Scalar Multiplication: for any $\alpha \in \mathbb{R}$ and $(x_1, x_2, x_3) \in \mathbf{U}$,

$$\alpha \cdot (x_1, x_2, x_3) = (x_1^\alpha, x_2^\alpha, x_3^\alpha).$$

(a) If $\mathbf{v} = (2, 3, 2)$, $\mathbf{w} = (1, 4, 5)$ and $\alpha = -1$, determine the vector $\mathbf{u} = \alpha \cdot (\mathbf{v} + \mathbf{w})$.

(b) Determine the zero vector $\mathbf{0}$ of $\mathbf{U}$.

(c) For any vector $\mathbf{v} = (x, y, z) \in \mathbf{U}$, find its additive inverse $-\mathbf{v}$.

5. **Uniqueness of the additive inverse:** In a (real) vector space $(\mathbf{V}, +, \cdot)$, show that any vector $\mathbf{v} \in \mathbf{V}$ has a unique (i.e., exactly one) additive inverse $-\mathbf{v}$.

Hint: To show that vector $\mathbf{v}$ has a unique additive inverse in $\mathbf{V}$, assume that it has two additive inverses, denoted by $-\mathbf{v}$ and $\bar{\mathbf{v}}$, respectively, and show that $-\mathbf{v} = \bar{\mathbf{v}}$.