Queen's University - APSC174 **Winter 2025**

Tutorial 02

1. Consider the set of all real-valued ordered n-tuples

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}$$

with $n \geq 2$. In class, we observed that $\mathbb{R}^n$ is a vector space under the following (component-wise) addition and scalar multiplication operations:

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$\alpha \cdot (x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n), \qquad \alpha \in \mathbb{R}.$$

Consider the subset $\mathbf{V}$ in $\mathbb{R}^n$ given by

$$\mathbf{V} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_1 + x_2 = 0\}$$

Is $\mathbf{V}$ a vector space under the above addition and scalar multiplication operations? Support your answer by referring to the 8 axioms of a vector space.

Solution. First recall that (as seen in class) the result that $\mathbb{R}^n$ is a vector space follows from the fact that the (component-wise) addition and scalar multiplication operations are valid in $\mathbb{R}^n$ (i.e., the addition of two vectors in $\mathbb{R}^n$ yields a vector in $\mathbb{R}^n$ and the scalar multiplication of any real number with a vector in $\mathbb{R}^n$ yields a vector in $\mathbb{R}^n$) and the fact that the 8 axioms of the vector space are satisfied. Recall that these axioms include the commutativity and associativity properties for vector addition, the existence of a zero vector (which is given here by $\mathbf{0} = (0, 0, \ . \ . \ . \ , 0)$, the all-zero tuple of length n), the existence of an additive inverse for each vector in $\mathbb{R}^n$, and four axioms that involve scalar multiplication as detailed in Section 2 of the Online Textbook.

Now, the above set $\mathbf{V}$ is itself a vector space under the addition and scalar multiplication operations of $\mathbb{R}^n$ (actually, it turns out that $\mathbf{V}$ is a vector subspace of $\mathbb{R}^n$, as will be seen in Section 3 of the Online Textbook). First, we verify that the above operations are valid in $\mathbf{V}$. For $\mathbf{v}_1 = (x_1, x_2, ..., x_n)$ and $\mathbf{v}_2 = (y_1, y_2, ..., y_n)$ in $\mathbf{V}$, we have that $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$; thus $\mathbf{v}_1 + \mathbf{v}_2 = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ with $(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 0 + 0 = 0$. Hence, $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbf{V}$ and the addition operation is valid in $\mathbf{V}$. Furthermore, for $\alpha \in \mathbb{R}$ and $\mathbf{v} = (x_1, x_2, ..., x_n) \in \mathbf{V}$, we have that $x_1 + x_2 = 0$; thus $\alpha \cdot \mathbf{v} = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$ with $\alpha x_1 + \alpha x_2 = \alpha(x_1 + x_2) = \alpha 0 = 0$. Hence $\alpha \cdot \mathbf{v} \in \mathbf{V}$ and the scalar multiplication operation is valid in $\mathbf{V}$.

We next need to verify that $\mathbf{V}$ satisfies the 8 vector space axioms. We will only verify the axiom about $\mathbf{V}$ having a zero vector. The other 7 axioms can be verified in the same way they were verified for $\mathbb{R}^n$ (actually since $\mathbf{V}$ is a subset of $\mathbb{R}^n$, these 7 axioms readily hold in $\mathbf{V}$). We next show that the zero vector $\mathbf{0} = (0, 0, ..., 0)$ of $\mathbb{R}^n$ is also the zero vector for $\mathbf{V}$. Indeed, since the sum of the first two components of $\mathbf{0} = (0, 0, ..., 0)$ is equal to zero ($0 + 0 = 0$), we directly obtain that $\mathbf{0} = (0, 0, ..., 0) \in \mathbf{V}$ and thus $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for any $\mathbf{v} \in \mathbf{V}$.

2. The set $\mathcal{C}^\infty(\mathbb{R})$, which is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are infinitely differentiable, is a vector space under the following addition and scalar multiplication operations: for any $f_1, f_2 \in \mathcal{C}^\infty(\mathbb{R})$,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad x \in \mathbb{R},$$

for any $\alpha \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(\mathbb{R})$,

$$(\alpha \cdot f)(x) = \alpha f(x), \quad x \in \mathbb{R}.$$

(a) What is the zero vector of $\mathcal{C}^\infty(\mathbb{R})$?

(b) For any vector in $\mathcal{C}^\infty(\mathbb{R})$, determine its additive inverse.

(c) Now consider the set $\mathbf{W}$ of all functions in $\mathcal{C}^\infty(\mathbb{R})$ that satisfy $f(7) = 1$:

$$\mathbf{W} = \{f \in \mathcal{C}^\infty(\mathbb{R}) : f(7) = 1\}.$$

Is $\mathbf{W}$ a vector space under the above addition and scalar multiplication operations?

Solution.
(a). The zero vector of $\mathcal{C}^\infty(\mathbb{R})$ is the zero function denoted by $\mathbf{0}(x)$, which maps every real number x to the zero real number (or zero scalar); i.e., $\mathbf{0} : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{0}(x) = 0$ for any $x \in \mathbb{R}$, that is,

$$\begin{aligned} \mathbf{0} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 0 \end{aligned}$$

First note that this function is infinitely differentiable and hence it belongs to $\mathcal{C}^\infty(\mathbb{R})$. Furthermore for any function $f \in \mathcal{C}^\infty(\mathbb{R})$, we have that for any $x \in \mathbb{R}$,

$$f(x) + \mathbf{0}(x) = f(x) + 0 = f(x)$$

and

$$\mathbf{0}(x) + f(x) = 0 + f(x) = f(x).$$

(b). For any $f \in \mathcal{C}^\infty(\mathbb{R})$, its additive inverse is given by the function $-f \in \mathcal{C}^\infty(\mathbb{R})$ such that $(-f)(x) = -f(x)$ for any $x \in \mathbb{R}$.
First note that since f is infinitely differentiable then so is $-f$; hence $-f \in \mathcal{C}^\infty(\mathbb{R})$. Furthermore, for any $x \in \mathbb{R}$,

$$(-f)(x) + f(x) = -f(x) + f(x) = 0 = \mathbf{0}(x)$$

and

$$f(x) + (-f)(x) = f(x) + (-f(x)) = f(x) - f(x) = 0 = \mathbf{0}(x).$$

(c). The set $\mathbf{W}$ (which is a subset of $\mathcal{C}^\infty(\mathbb{R})$) is not a vector space under the operations of $\mathcal{C}^\infty(\mathbb{R})$ since the addition operation is not well-defined in $\mathbf{W}$. Indeed for any f and g in $\mathbf{W}$, we have that $f(7) = 1$ and $g(7) = 1$; but

$$(f + g)(7) = f(7) + g(7) = 1 + 1 = 2 \neq 1$$

and hence $f + g \notin \mathbf{W}$. Note also that the zero vector $\mathbf{0}(\cdot)$ of $\mathcal{C}^\infty(\mathbb{R})$ does not belong to $\mathbf{W}$ since $\mathbf{0}(7) = 0 \neq 1$.

3. Consider the set of all real-valued pairs $\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ under the following **new** addition and scalar multiplication operations:

Addition: for any (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2$,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 - 1, x_2 + y_2 - 2).$$

Scalar multiplication: for any $(x_1, x_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$,

$$\alpha \cdot (x_1, x_2) = (\alpha x_1 - \alpha + 1, \alpha x_2 - 2\alpha + 2).$$

Show that $\mathbb{R}^2$ is a vector space under the above operations by demonstrating that each of the 8 axioms of a vector space is satisfied.

Solution. First note that above new addition and scalar multiplication operations are valid in $\mathbb{R}^2$ (verify from the above definitions of addition and scalar multiplication that $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{R}^2$ for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and that $\alpha\mathbf{v} \in \mathbb{R}^2$ for any $\alpha \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$).

We next show that $\mathbb{R}^2$ satisfies the 8 vector space axioms under these **new** addition and scalar multiplication operations:

i. *Associativity of vector addition* $(+)$: For any $\mathbf{u} = (x_1, x_2)$, $\mathbf{v} = (y_1, y_2)$ and $\mathbf{w} = (z_1, z_2)$ in $\mathbb{R}^2$,

$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) \\ &= (x_1, x_2) + (y_1 + z_1 - 1, y_2 + z_2 - 2) \\ &= (x_1 + (y_1 + z_1 - 1) - 1, x_2 + (y_2 + z_2 - 2) - 2) \\ &= (x_1 + y_1 + z_1 - 2, x_2 + y_2 + z_2 - 4).\end{aligned}$$

Also,

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) \\ &= (x_1 + y_1 - 1, x_2 + y_2 - 2) + (z_1, z_2) \\ &= ((x_1 + y_1 - 1) + z_1 - 1, (x_2 + y_2 - 2) + z_2 - 2) \\ &= (x_1 + y_1 + z_1 - 2, x_2 + y_2 + z_2 - 4).\end{aligned}$$

Thus $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

ii. *Zero vector*: We show that the zero vector of $\mathbb{R}^2$ under the above operations is given by $\mathbf{0} = (1, 2)$. Indeed, we have that for any $\mathbf{u} = (x_1, x_2) \in \mathbb{R}^2$,

$$\mathbf{u} + (1, 2) = (x_1, x_2) + (1, 2) = (x_1 + 1 - 1, x_2 + 2 - 2) = (x_1, x_2) = \mathbf{u}$$

and

$$(1, 2) + \mathbf{u} = (1, 2) + (x_1, x_2) = (1 + x_1 - 1, 2 + x_2 - 2) = (x_1, x_2) = \mathbf{u}.$$

Thus the zero vector is $\mathbf{0} = (1, 2)$.

iii. *Additive inverse*: We show that for any $\mathbf{u} = (x_1, x_2) \in \mathbb{R}^2$, its additive inverse is given by $-\mathbf{u} = (-x_1 + 2, -x_2 + 4)$:

$$\begin{aligned}\mathbf{u} + (-x_1 + 2, -x_2 + 4) &= (x_1, x_2) + (-x_1 + 2, -x_2 + 4) \\ &= (x_1 - x_1 + 2 - 1, x_2 - x_2 + 4 - 2) \\ &= (1, 2) = \mathbf{0}\end{aligned}$$

and

$$\begin{aligned}(-x_1 + 2, -x_2 + 4) + \mathbf{u} &= (-x_1 + 2, -x_2 + 4) + (x_1, x_2) \\ &= (-x_1 + 2 + x_1 - 1, -x_2 + 4 + x_2 - 2) \\ &= (1, 2) = \mathbf{0}.\end{aligned}$$

Thus the additive inverse of $\mathbf{u} = (x_1, x_2)$ is $-\mathbf{u} = (-x_1 + 2, -x_2 + 4)$.

iv. *Commutativity of* $+$: For any $\mathbf{u} = (x_1, x_2)$ and $\mathbf{v} = (y_1, y_2)$ in $\mathbb{R}^2$,

$$\mathbf{u} + \mathbf{v} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1 - 1, x_2 + y_2 - 2)$$

and

$$\mathbf{v} + \mathbf{u} = (y_1, y_2) + (x_1, x_2) = (y_1 + x_1 - 1, y_2 + x_2 - 2) = (x_1 + y_1 - 1, x_2 + y_2 - 2).$$

Thus $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

vi. *Distributivity of $\cdot$ with respect to $+$*: For any $\alpha \in \mathbb{R}$ and $\mathbf{u} = (x_1, x_2)$ and $\mathbf{v} = (y_1, y_2)$ in $\mathbb{R}^2$,

$$\begin{aligned}
\alpha \cdot (\mathbf{u} + \mathbf{v}) &= \alpha \cdot ((x_1, x_2) + (y_1, y_2)) \\
&= \alpha \cdot (x_1 + y_1 - 1, x_2 + y_2 - 2) \\
&= (\alpha(x_1 + y_1 - 1) - \alpha + 1, \alpha(x_2 + y_2 - 2) - 2\alpha + 2) \\
&= (\alpha(x_1 + y_1) - 2\alpha + 1, \alpha(x_2 + y_2) - 4\alpha + 2)
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha \cdot \mathbf{u}) + (\alpha \cdot \mathbf{v}) \\
&= (\alpha \cdot (x_1, x_2)) + (\alpha \cdot (y_1, y_2)) \\
&= (\alpha x_1 - \alpha + 1, \alpha x_2 - 2\alpha + 2) + (\alpha y_1 - \alpha + 1, \alpha y_2 - 2\alpha + 2) \\
&= ((\alpha x_1 - \alpha + 1) + (\alpha y_1 - \alpha + 1) - 1, (\alpha x_2 - 2\alpha + 2) + (\alpha y_2 - 2\alpha + 2) - 2) \\
&= (\alpha(x_1 + y_1) - 2\alpha + 1, \alpha(x_2 + y_2) - 4\alpha + 2).
\end{aligned}$$

Thus $\alpha \cdot (\mathbf{u} + \mathbf{v}) = (\alpha \cdot \mathbf{u}) + (\alpha \cdot \mathbf{v})$.

vii. *Distributivity of $\cdot$ with respect to scalar addition*: For any α and β in $\mathbb{R}$ and any $\mathbf{u} = (x_1, x_2)$ in $\mathbb{R}^2$,

$$\begin{aligned}
&(\alpha + \beta) \cdot \mathbf{u} = (\alpha + \beta) \cdot (x_1, x_2) \\
&= ((\alpha + \beta)x_1 - (\alpha + \beta) + 1, (\alpha + \beta)x_2 - 2(\alpha + \beta) + 2)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u} = \alpha \cdot (x_1, x_2) + \beta \cdot (x_1, x_2) \\
&= (\alpha x_1 - \alpha + 1, \alpha x_2 - 2\alpha + 2) + (\beta x_1 - \beta + 1, \beta x_2 - 2\beta + 2) \\
&= (\alpha x_1 - \alpha + 1 + \beta x_1 - \beta + 1 - 1, \alpha x_2 - 2\alpha + 2 + \beta x_2 - 2\beta + 2 - 2) \\
&= ((\alpha + \beta)x_1 - (\alpha + \beta) + 1, (\alpha + \beta)x_2 - 2(\alpha + \beta) + 2).
\end{aligned}$$

Thus $(\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$.

viii. *Property of scalar identity*: For any $\mathbf{u} = (x_1, x_2)$ in $\mathbb{R}^2$,

$$1 \cdot \mathbf{u} = 1 \cdot (x_1, x_2) = ((1)(x_1) - 1 + 1, (1)(x_2) - 2(1) + 2) = (x_1, x_2) = \mathbf{u}.$$

Thus $1 \cdot \mathbf{u} = \mathbf{u}$.

We conclude that $\mathbb{R}^2$ is a vector space axioms under the above **new** addition and scalar multiplication operations.

4. Consider the vector space

$$\mathbf{U} = \{(x, y, z) : x, y, z \in \mathbb{R}, x > 0, y > 0, z > 0\}$$

under the following addition and scalar multiplication operations:

Addition: for any (x_1, x_2, x_3) and (y_1, y_2, y_3) in $\mathbf{U}$,

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 y_1, x_2 y_2, x_3 y_3).$$

Scalar Multiplication: for any $\alpha \in \mathbb{R}$ and $(x_1, x_2, x_3) \in \mathbf{U}$,

$$\alpha \cdot (x_1, x_2, x_3) = (x_1^\alpha, x_2^\alpha, x_3^\alpha).$$

(a) If $\mathbf{v} = (2, 3, 2)$, $\mathbf{w} = (1, 4, 5)$ and $\alpha = -1$, determine the vector $\mathbf{u} = \alpha \cdot (\mathbf{v} + \mathbf{w})$.

(b) Determine the zero vector $\mathbf{0}$ of $\mathbf{U}$.

(c) For any vector $\mathbf{v} = (x, y, z) \in \mathbf{U}$, find its additive inverse $-\mathbf{v}$.

Solution.
(a). We have

$$\begin{aligned}
\mathbf{u} &= \alpha \cdot (\mathbf{v} + \mathbf{w}) \\
&= (-1) \cdot ((2, 3, 2) + (1, 4, 5)) \\
&= (-1) \cdot ((2)(1), (3)(4), (2)(5)) \\
&= (-1) \cdot (2, 12, 10) \\
&= (2^{-1}, 12^{-1}, 10^{-1}) \\
&= \left(\frac{1}{2}, \frac{1}{12}, \frac{1}{10}\right)
\end{aligned}$$

(b). The zero vector of $\mathbf{U}$ is given by $\mathbf{0} = (1, 1, 1)$, because for any $(x_1, x_2, x_3) \in \mathbf{U}$ we have

$$\mathbf{0} = 0 \cdot (x_1, x_2, x_3) = (x_1^0, x_2^0, x_3^0) = (1, 1, 1).$$

We verify the zero vector as follows. First note that $(1, 1, 1) \in \mathbf{U}$. Furthermore, for any $\mathbf{v} = (x, y, z) \in \mathbf{U}$,

$$\mathbf{v} + (1, 1, 1) = (x, y, z) + (1, 1, 1) = ((x)(1), (y)(1), (z)(1)) = (x, y, z) = \mathbf{v}$$

and

$$(1, 1, 1) + \mathbf{v} = (1, 1, 1) + (x, y, z) = ((1)(x), (1)(y), (1)(z)) = (x, y, z) = \mathbf{v}.$$

Thus $\mathbf{0} = (1, 1, 1)$.

(c). For any vector $\mathbf{v} = (x, y, z) \in \mathbf{U}$, its additive inverse is given by $-\mathbf{v} = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$, because

$$-\mathbf{v} = (-1) \cdot \mathbf{v} = (-1) \cdot (x, y, x) = (x^{-1}, y^{-1}, z^{-1}) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right).$$

We next verify this inverse additive as follows. First note that $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \in \mathbf{U}$ since $x > 0$, $y > 0$ and $z > 0$. Furthermore, for any $\mathbf{v} = (x, y, z) \in \mathbf{U}$,

$$\begin{aligned}
\mathbf{v} + \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) &= (x, y, z) + \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \\
&= \left(\frac{x}{x}, \frac{y}{y}, \frac{z}{z}\right) \\
&= (1, 1, 1) \\
&= \mathbf{0}
\end{aligned}$$

and similarly, $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) + \mathbf{v} = (1, 1, 1) = \mathbf{0}$. Hence,

$$-\mathbf{v} = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right).$$

5. **Uniqueness of the additive inverse:** In a (real) vector space $(\mathbf{V}, +, \cdot)$, show that any vector $\mathbf{v} \in \mathbf{V}$ has a unique (i.e., exactly one) additive inverse $-\mathbf{v}$.

Hint: To show that vector $\mathbf{v}$ has a unique additive inverse in $\mathbf{V}$, assume that it has two additive inverses, denoted by $-\mathbf{v}$ and $\bar{\mathbf{v}}$, respectively, and show that $-\mathbf{v} = \bar{\mathbf{v}}$.

Solution. Given vector space $(\mathbf{V}, +, \cdot)$ and the vector $\mathbf{v} \in \mathbf{V}$, assume that $-\mathbf{v}$ and $\bar{\mathbf{v}}$ are additive inverses of $\mathbf{v}$; i.e, there exist vectors $-\mathbf{v}$ and $\bar{\mathbf{v}}$ in $\mathbf{V}$ that satisfy

$$\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0} \tag{1}$$

and

$$\mathbf{v} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \mathbf{v} = \mathbf{0} \tag{2}$$

respectively, where $\mathbf{0}$ denotes the zero vector of $\mathbf{V}$. We thus have

$$\begin{aligned}
-\mathbf{v} &= -\mathbf{v} + \mathbf{0} && \text{(by the property of } \mathbf{0} \text{ (Axiom 2 of a vector space))} \\
&= -\mathbf{v} + (\mathbf{v} + \bar{\mathbf{v}}) && \text{(by (2) above)} \\
&= (-\mathbf{v} + \mathbf{v}) + \bar{\mathbf{v}} && \text{(by associativity of addition in } \mathbf{V} \text{ (Axiom 1 of a vector space))} \\
&= \mathbf{0} + \bar{\mathbf{v}} && \text{(by (1) above)} \\
&= \bar{\mathbf{v}} && \text{(by the property of } \mathbf{0} \text{ (Axiom 2 of a vector space.))}
\end{aligned}$$

Thus $-\mathbf{v} = \bar{\mathbf{v}}$ and the additive inverse of $\mathbf{v}$ is unique.