Queen's University - APSC174 Winter 2025

Tutorial 03

Recall that a subset $\mathbf{W}$ is a vector subspace of a vector space $\mathbf{V}$ if $\mathbf{W}$ satisfies the following three axioms:

(i) The zero vector in $\mathbf{V}$ is an element of $\mathbf{W}$, i.e., $\mathbf{0} \in \mathbf{W}$.

(ii) If $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$, then $\mathbf{w}_1 + \mathbf{w_2} \in \mathbf{W}$.

(iii) If $\mathbf{w} \in \mathbf{W}$ and $\alpha \in \mathbb{R}$, then $\alpha \cdot \mathbf{w} \in \mathbf{W}$.

Hence in order to check whether or not a subset of $\mathbf{V}$ is a subspace of $\mathbf{V}$, we have to see if this subset passes all three conditions for the following 3 problems.

1. Consider the usual vector space $(\mathbb{R}^2, +, \cdot)$. For each of the following subsets of $\mathbb{R}^2$, determine whether or not it is a vector subspace of $\mathbb{R}^2$:

(a) $S =$ set of all (x, y) in $\mathbb{R}^2$ such that $6x + 8y = 0$.

(b) $S =$ set of all (x, y) in $\mathbb{R}^2$ such that $6x + 8y = 1$.

Solution.

(a) In order to check whether or not S is a subspace of $\mathbb{R}^2$, we have to see if S passes all three conditions. Set S is the collection of all the points on the line $6x+8y = 0$, which passes through the origin $(0, 0)$.

(i) Since the zero vector $\mathbf{0} = (0, 0)$ (that is, point $(0, 0)$) in $\mathbb{R}^2$ is on the line $6x + 8y = 0$, we have $\mathbf{0} = (0, 0) \in S$.

(ii) Let $\mathbf{w}_1 = (x_1, y_1), \mathbf{w}_2 = (x_2, y_2) \in S$, that is,

$$6x_1 + 8y_1 = 0, \quad 6x_2 + 8y_2 = 0.$$

Then,

$$\mathbf{w}_1 + \mathbf{w}_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

and

$$6(x_1 + x_2) + 8(y_1 + y_2) = 6x_1 + 8y_1 + 6x_2 + 8y_2 = 0 + 0 = 0$$

Therefore, point $(x_1 + x_2, y_1 + y_2)$ is on the line $6x + 8y = 0$, that is, $\mathbf{w}_1 + \mathbf{w}_2 \in S$.

(iii) Let $\alpha \in \mathbb{R}$, $\mathbf{w} = (x, y) \in S$, that is,

$$6x + 8y = 0.$$

Then,

$$\alpha \cdot \mathbf{w} = \alpha \cdot (x, y) = (\alpha x, \alpha y)$$

and

$$6(\alpha x) + 8(\alpha y) = \alpha(6x + 8y) = \alpha(0) = 0.$$

Therefore, point $(\alpha x, \alpha y)$ is on the line $6x + 8y = 0$, that is, $\alpha \cdot \mathbf{w} \in S$.

We now have proved that S satisfies all three conditions, so it is a subspace of $\mathbb{R}^2$.

(b) In order to check whether or not S is a subspace of $\mathbb{R}^2$, we have to see if S passes all three conditions. Set S is the collection of all the points on the line $6x+8y=1$. Since the zero vector $\mathbf{0}=(0,0)$ (that is, point $(0,0)$) in $\mathbb{R}^2$ is not on the line $6x+8y=1$, the zero vector is not in the subset S, which implies that this subset in part (b) is not a subspace of $\mathbb{R}^2$.

2. Recall that $\mathcal{F}(\mathbb{R};\mathbb{R})$ denotes the vector space of functions $f:\mathbb{R}\to\mathbb{R}$ with the usual function addition and scalar multiplication. For each of the following subsets of $\mathcal{F}(\mathbb{R};\mathbb{R})$, determine whether or not it is a vector subspace of $\mathcal{F}(\mathbb{R};\mathbb{R})$:

(a) $S=$ set of all f in $\mathcal{F}(\mathbb{R};\mathbb{R})$ such that $f(x)+f(x+1)+f(x+2)=1$ for all $x\in\mathbb{R}$.

(b) $S=$ set of all f in $\mathcal{F}(\mathbb{R};\mathbb{R})$ such that $f(x)+f(x+1)+f(x+2)=0$ for all $x\in\mathbb{R}$.

Solution. In order to check whether or not S is a subspace of $\mathcal{F}(\mathbb{R};\mathbb{R})$, we have to see if S passes all three conditions.

(a) Since the zero vector of $\mathcal{F}(\mathbb{R};\mathbb{R})$ is the zero function $\mathbf{0}(x)=0$ for any $x\in\mathbb{R}$, we have

$$\mathbf{0}(x)+\mathbf{0}(x+1)+\mathbf{0}(x+2)=0\neq 1.$$

Hence, the zero vector is not in S, that is, the subset S in part (a) is not a subspace of $\mathcal{F}(\mathbb{R};\mathbb{R})$.

(b) We check all three conditions for S.

(i) Since

$$\mathbf{0}(x)+\mathbf{0}(x+1)+\mathbf{0}(x+2)=0$$

for all $x\in\mathbb{R}$, we have $\mathbf{0}\in\mathcal{F}(\mathbb{R};\mathbb{R})$.

(ii) Let $f,g\in\mathcal{F}(\mathbb{R};\mathbb{R})$, and we have

$$f(x)+f(x+1)+f(x+2)=0$$

and

$$g(x)+g(x+1)+g(x+2)=0$$

for any $x\in\mathbb{R}$, then

$$\begin{aligned}
&(f+g)(x)+(f+g)(x+1)+(f+g)(x+2)\\
&=f(x)+g(x)+f(x+1)+g(x+1)+f(x+2)+g(x+2)\\
&=(f(x)+f(x+1)+f(x+2))+(g(x)+g(x+1)+g(x+2))\\
&=0+0\\
&=0
\end{aligned}$$

which implies that $f+g\in S$.

(iii) Let $\alpha\in\mathbb{R}$ and $f\in\mathcal{F}(\mathbb{R};\mathbb{R})$, and we have

$$f(x)+f(x+1)+f(x+2)=0,$$

then

$$(\alpha\cdot f)(x)=\alpha f(x)$$

and

$$
\begin{aligned}
&(\alpha \cdot f)(x) + (\alpha \cdot f)(x+1) + (\alpha \cdot f)(x+2) \\
&= \alpha f(x) + \alpha f(x+1) + \alpha f(x+2) \\
&= \alpha(f(x) + f(x+1) + f(x+2)) \\
&= \alpha \cdot 0 \\
&= 0
\end{aligned}
$$

that is, $\alpha \cdot f \in S$.

We now have proved that S satisfies all three conditions, so it is a subspace of $f \in \mathcal{F}(\mathbb{R};\mathbb{R})$.

3. Consider the following vector space

$$\mathbf{W}_2 = \{(x_1, x_2) : \ x_1, x_2 \in \mathbb{R}, \ x_1 > 0, \ x_2 > 0\}$$

under the following operations:

Addition: for any $\mathbf{u} = (x_1, x_2)$ and $\mathbf{v} = (y_1, y_2)$ in $\mathbf{W}_2$,

$$\mathbf{u} + \mathbf{v} = (x_1, x_2) + (y_1, y_2) = (x_1 y_1, x_2 y_2)$$

Scalar multiplication: for any $\alpha \in \mathbb{R}$ and $\mathbf{u} = (x_1, x_2) \in \mathbf{W}_2$,

$$\alpha \cdot \mathbf{u} = \alpha \cdot (x_1, x_2) = (x_1^{\alpha}, x_2^{\alpha}) \, .$$

(a) Determine which of the following subsets of $\mathbf{W}_2$ is a subspace of $\mathbf{W}_2$:

(a1) $S =$ set of all (x_1, x_2) in $\mathbf{W}_2$ such that $x_1 x_2 = 0$.

(a2) $S =$ set of all (x_1, x_2) in $\mathbf{W}_2$ such that $x_1^2 x_2 = 1$.

(b) Consider the following vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbf{W}_2$ defined as follows:

$$\mathbf{v}_1 = (1, 2), \quad \mathbf{v}_2 = (2, 1), \quad \mathbf{v}_3 = (3, 2).$$

(b1) Is $\mathbf{v}_1$ in the linear span of $\{\mathbf{v}_2\}$?

(b2) Is $\mathbf{v}_3$ in the linear span of $\{\mathbf{v}_1, \mathbf{v}_2\}$?

Solution.

(a) In order to check whether or not S is a subspace of $\mathbf{W}_2$, we have to see if S passes all three conditions. From the scalar multiplication for $\mathbf{W}_2$, we can find its zero vector as follows:

$$\mathbf{0} = 0 \cdot \mathbf{v} = 0 \cdot (x_1, x_2) = (x_1^0, x_2^0) = (1, 1),$$

that is, $\mathbf{0} = (1, 1)$.

(a1) Let $(x_1, x_2) = (1, 1)$, then $x_1 x_2 = 1 \cdot 1 = 1 \neq 0$, which implies $(1, 1) \notin S$, that is, $\mathbf{0} \notin S$. Thus, this subset is not a subspace of $\mathbf{W}_2$. Actually, this subset of $\mathbf{W}_2$ is an empty set.

(a2) (i) Let $(x_1, x_2) = (1, 1)$, then $x_1 x_2 = 1 \cdot 1$, which implies $(1, 1) \in S$, that is, $\mathbf{0} \in S$.

(ii) Let $\mathbf{w}_1 = (x_1, x_2), \mathbf{w}_2 = (y_1, y_2) \in \mathbf{W}_2$, then we have

$$x_1^2 x_2 = 1 \text{ and } y_1^2 y_2 = 1.$$

From the addition operation, we have

$$(x_1, x_2) + (y_1, y_2) = (x_1 y_1, x_2 y_2)$$

and then

$$(x_1 y_1)^2 (x_2 y_2) = (x_1^2 x_2)(y_1^2 y_2) = 1 \cdot 1 = 1,$$

that is, $(x_1 y_1, x_2 y_2) \in S$. Therefore, $\mathbf{w}_1 + \mathbf{w}_2 \in S$.

(iii) Let $\alpha \in \mathbb{R}$ and $\mathbf{w} = (x_1, x_2) \in \mathbf{W}_2$, then we have

$$x_1^2 x_2 = 1 \text{ and } \alpha \cdot \mathbf{w} = \alpha \cdot (x_1, x_2) = (x_1^\alpha, x_2^\alpha)$$

which implies

$$\left(x_1^\alpha\right)^2 \left(x_2^\alpha\right) = \left(x_1^2\right)^\alpha \left(x_2^\alpha\right) = \left(x_1^2 x_2\right)^\alpha = 1^\alpha = 1,$$

that is, $(x_1^\alpha, x_2^\alpha) \in S$. Hence, $\alpha \cdot \mathbf{w} \in S$.

We now have proved that S satisfies all three conditions, so it is a subspace of $\mathbf{w} \in S$.

(b) This part focuses on linear spans of $\mathbf{W}_2$.

(b1) We are asked if $\mathbf{v}_1$ can be written as a linear combination of $\mathbf{v}_2$, i.e., if there exist $\alpha \in \mathbb{R}$ such that

$$\mathbf{v}_1 = \alpha \cdot \mathbf{v}_2.$$

This equation becomes

$$(1, 2) = \alpha \cdot (2, 1)$$

which, after applying the rules for scalar multiplication and addition of vectors in $\mathbf{W}_2$, gives the single vector equation

$$(1, 2) = (2^\alpha, 1^\alpha).$$

Matching the corresponding components of the vectors on the left and right hand sides, we obtain

$$2^\alpha = 1 \text{ and } 1^\alpha = 2.$$

The first equation implies $\alpha = 0$ but $1^\alpha = 1^0 = 1 \neq 2$. Hence, there does not exists an α so that the above two equations are satisfied simultaneously, that is, there is no such α so that $\mathbf{v}_1 = \alpha \cdot \mathbf{v}_2$.

Therefore, $\mathbf{v}_1$ is not in the linear span of $\{\mathbf{v}_2\}$.

(b2) We are asked if $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e., if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{v}_3 = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2.$$

This equation becomes

$$(3, 2) = \alpha \cdot (1, 2) + \beta \cdot (2, 1).$$

which, after applying the rules for scalar multiplication and addition of vectors in $\mathbf{W}_2$, gives the single vector equation

$$(3, 2) = (1^\alpha, 2^\alpha) + (2^\beta, 1^\beta) = (1^\alpha 2^\beta, 2^\alpha 1^\beta) = (2^\beta, 2^\alpha).$$

Matching the corresponding components of the vectors on the left and right hand sides, we obtain

$$2^\beta = 3 \text{ and } 2^\alpha = 2.$$

which imply $\beta = \frac{\ln(3)}{\ln(2)}$ and $\alpha = 1$. Hence,

$$\mathbf{v}_3 = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2 = 1 \cdot \mathbf{v}_1 + \frac{\ln(3)}{\ln(2)} \cdot \mathbf{v}_2,$$

that is, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Therefore, $\mathbf{v}_3$ is in the linear span of $\{\mathbf{v}_1\}$ and $\{\mathbf{v}_2\}$.

4. Consider $\mathbb{R}^3$ with the usual (component-wise) addition and scalar multiplication operations. Show that the linear span of the vectors $(1,0,0)$, $(1,1,0)$, $(0,1,1)$ is $\mathbb{R}^3$ itself.

Solution. Let $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (1,1,0)$, $\mathbf{v}_3 = (0,1,1)$. We have to show that $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} = \mathbb{R}^3$. To show equality between two sets, our strategy is to show both that

(i) $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} \subset \mathbb{R}^3$ and

(ii) $\mathbb{R}^3 \subset S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$.

Since all the vectors defining $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$ are elements of $\mathbb{R}^3$, we have every linear combination of them is also in $\mathbb{R}^3$, that is, every element of $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$ is also in $\mathbb{R}^3$. Hence, $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} \subset \mathbb{R}^3$. We showed (i) is true.

Next, we show (ii): $\mathbb{R}^3 \subset S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$. To do this, we need to show that for any $\mathbf{w} = (x,y,z) \in \mathbb{R}^3$ there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3 = \mathbf{w},$$

or equivalently,

$$\alpha(1,0,0) + \beta(1,1,0) + \gamma(0,1,1) = (x,y,z).$$

This gives the single vector equation

$$(\alpha+\beta, \beta+\gamma, \gamma) = (x,y,z).$$

As before, this vector equation is equivalent to the system of three linear equations in unknowns α, β, γ:

$$\alpha + \beta = x \tag{1}$$
$$\beta + \gamma = y \tag{2}$$
$$\gamma = z \tag{3}$$

We can solve this system by any process you have seen for solving linear equations. Here, we will add and subtract equations. From (3) and (2), we have

$$\beta + \gamma = \beta + z = y \Rightarrow \beta = y - z \tag{4}$$

Then (4) and (1) imply that

$$\alpha + \beta = \alpha + (y-z) = x \Rightarrow \alpha = x - (y-z) \tag{5}$$

Therefor, given any $\mathbf{w} = (x,y,z) \in \mathbb{R}^3$ we can find scalars α, β, γ:

$$\begin{aligned} \alpha &= x - y + z \\ \beta &= y - z \\ \gamma &= z \end{aligned}$$

so that

$$\alpha(1,0,0) + \beta(1,1,0) + \gamma(0,1,1) = (x,y,z).$$

This shows that every vector in $\mathbb{R}^3$ is in the span of $(1,0,0)$, $(1,1,0)$ and $(0,1,1)$, i.e., $\mathbb{R}^3 \subset S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$.

With both $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} \subset \mathbb{R}^3$ and $\mathbb{R}^3 \subset S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$, we can conclude that $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} = \mathbb{R}^3$.