1. In this problem we consider the vector space $\mathbb{R}^n$ of all real-valued column n-tuples, for $n = 3$ and $n = 4$, with the usual (component-wise) addition and scalar multiplication operations. For each of the cases below, determine whether or not the vector $\mathbf{w}$ can be expressed as a linear combination of the other two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$.

(a) $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(b) $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

2. (Problem 4 in Tutorial 03) Consider $\mathbb{R}^3$ with the usual (component-wise) addition and scalar multiplication operations. Show that the linear span of the vectors $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 1)$ is $\mathbb{R}^3$ itself.

3. Define $P_2(\mathbb{R}) \subset C^\infty(\mathbb{R})$, or more simply P_2, as the set of all real polynomials of degree 2 or less; i.e.,

$$P_2(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f(x) = a_0 + a_1 x + a_2 x^2, x \in \mathbb{R} \text{ for some real } a_0, a_1, a_2\}.$$

Show that the linear span of the vectors (or functions) $f_1(x) = 1$, $f_2(x) = 1 + x$, and $f_3(x) = x + x^2$, where $x \in \mathbb{R}$, is P_2 itself.

4. If you compare your solutions to Questions 2 and 3 you will note some similarities. From the perspective of vector spaces (adding and scalar multiplying elements), how can you describe the relationship between $\mathbb{R}^3$ and P_2?

5. Let $\mathbf{V}$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in \mathbf{V}$ ($n \geqslant 2$). Show that if $S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)} = S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1})}$, then $\mathbf{v}_n \in S_{(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1})}$.

6. In class we have seen that if $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then we can convert the question "Is $\mathbf{w}$ a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$?" into a question "Does this system of linear equations have a solution?".

(a) Let $\mathbf{w} = (1, 3, 5, 3)$, $\mathbf{v}_1 = (2, 4, 3, 0)$, $\mathbf{v}_2 = (1, 1, -2, -1)$, $\mathbf{v}_3 = (0, 2, 7, 3)$. What system of linear equations does the question "Is $\mathbf{w}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?" correspond to?

(NOTE: You do not have to solve the system of linear equations — the purpose of this question is just to write them down.)

(b) Conversely, consider the system of linear equations below :

$$\begin{array}{rcrcrcrcl} 2x_1 & - & x_2 & + & 4x_3 & + & 9x_4 & = & 0 \\ x_1 & + & 5x_2 & + & x_3 & - & 4x_4 & = & 8 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 1. \end{array}$$

Which "linear combination" problem does this system correspond to?

7. Let

$$\mathbf{W} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ 3x - 2y + z = 0 \right\}.$$

(a) Show, with proof, that $\mathbf{W}$ is a subspace of $\mathbb{R}^3$.

(b) Let $\mathbf{v}_1 = (1, 0, -3)$ and $\mathbf{v}_2 = (0, 1, 2)$. Check that $\mathbf{v}_1,\ \mathbf{v}_2 \in \mathbf{W}$.

(c) Show that $S_{(\mathbf{v}_1, \mathbf{v}_2)} \subset \mathbf{W}$.

(d) Check that $(4, 5, -2) \in \mathbf{W}$.

(e) Write $(4, 5, -2)$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(f) Show that $\mathbf{W} \subset S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

(g) Show that $\mathbf{W} = S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

8. For each of the following sets of vectors, determine if they are linearly dependent or linearly independent.

(a) $\{(1, 2),\ (5, 1), (1, 0)\} \subset \mathbb{R}^2$.

(b) $\{(2, 1, 3), (-1, 2, 1),\ (0, 4, 5)\} \subset \mathbb{R}^3$.

(c) $\{(2, 0, 3), (-1, 2, 1),\ (0, 4, 5)\} \subset \mathbb{R}^3$.

(d) $\{(1, 2, 4),\ (0, 0, 0),\ (2, 3, 1)\} \subset \mathbb{R}^3$.

(e) $\{\sin(x),\ \cos(x),\ 1\} \subset C^\infty(\mathbb{R})$.

(f) $\{\sin^2(x),\ \cos^2(x),\ 1\} \subset C^\infty(\mathbb{R})$.

(g) $\{1,\ x,\ x^2\} \subset C^\infty(\mathbb{R})$.

(h) $\{(2, 3),\ (3, 5)\} \subset \mathbf{W}_2$

(i) $\{(2, 3),\ (1, 1)\} \subset \mathbf{W}_2$

NOTES FOR PROBLEM 8:

(1) When we write something like "1"for an element of $C^\infty(\mathbb{R})$ we mean the constant function 1. That is, the function $f(x)$ defined by $f(x) = 1$ for all $x \in \mathbb{R}$. Similarly for any other number.

(2) When trying to decide of a set of functions is linearly dependent or linearly independent, as in (e)–(g), one way to go from an equation like $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$ to equations with numbers is to plug in different values of x.

(3) In parts (h) and (i) of question 8 the space $\mathbf{W}_2$ is the "weird" vector space introduced in class. Namely, as a set $\mathbf{W}_2 = \{(x, y) \ : \ x, y \in \mathbb{R},\ x, y > 0\}$ with operations $(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1, y_2)$ and $\alpha \cdot (x, y) = (x^\alpha, y^\alpha)$.