1. In this problem we consider the vector space $\mathbb{R}^n$ of all real-valued column n-tuples, for $n = 3$ and $n = 4$, with the usual (component-wise) addition and scalar multiplication operations. For each of the cases below, determine whether or not the vector $\mathbf{w}$ can be expressed as a linear combination of the other two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$.

(a) $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(b) $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Solution.

(a) We are asked if $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$ can be written as a linear combination of $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, i.e., if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{w}.$$

The equation becomes

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$$

which, after applying the rules for scalar multiplication and addition of vectors in $\mathbb{R}^4$, gives the single vector equation

$$\begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \\ -\alpha + \beta \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$$

Matching the corresponding components of the vectors on the left and right hand sides, we obtain the system of linear equations with unknowns α and β:

$$\begin{array}{rcr} \alpha & = & 2 \\ \beta & = & -1 \\ \alpha + \beta & = & 1 \\ -\alpha + \beta & = & -3 \end{array}$$

Since this system of equations is equivalent to the original vector equation for the linear combination, it has a solution exactly when $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. From the first two equations we obtain $\alpha = 2$ and $\beta = -1$. We then must check by substitution that these values also solve the third and fourth equations, which they do:

$$\alpha + \beta = 2 + (-1) = 1$$
$$-\alpha + \beta = -2 + (-1) = -3$$

Thus $\mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$ can be written as a linear combination of $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ since

$$\underbrace{\begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}}_{\mathbf{w}} = \underbrace{2}_{\alpha} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}}_{\mathbf{v}_1} + \underbrace{(-1)}_{\beta} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{\mathbf{v}_2}.$$

(b) Here we are asked if $\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ can be written as a linear combination of $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, i.e., whether there are $\alpha, \beta \in \mathbb{R}$ with

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{w}$$

or equivalently,

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

which gives the single vector equation

$$\begin{pmatrix} \alpha - \beta \\ \alpha + \beta \\ 2\alpha + \beta \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Matching again the corresponding entries of the vectors on the left and right hand sides, we obtain the system of linear equations with unknowns α and β:

$$\begin{aligned} \alpha - \beta &= -1 \\ \alpha + \beta &= 2 \\ 2\alpha + \beta &= 1 \end{aligned}$$

As in part (a), this system of linear equations is equivalent to the original vector equation for the linear combination; it has a solution exactly when $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Adding the first equation to the second one we get $2\alpha = 1$, i.e., $\alpha = 1/2$. Subtracting the first equation from the second one we get $2\beta = 3$, i.e., $\beta = 3/2$. But then $2\alpha + \beta = 2 \cdot (1/2) + (3/2) = 5/2$, so this α and β violate the third equation.

This means that there exist no α and β that simultaneously solve all three equations, i.e., this system of linear equation has no solution. Thus $\mathbf{w}$ **cannot** be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

2. (Problem 4 in Tutorial 03) Consider $\mathbb{R}^3$ with the usual (component-wise) addition and scalar multiplication operations. Show that the linear span of the vectors $(1,0,0), (1,1,0), (0,1,1)$ is $\mathbb{R}^3$ itself.

3. Define $P_2(\mathbb{R}) \subset C^\infty(\mathbb{R})$, or more simply P_2, as the set of all real polynomials of degree 2 or less; i.e.,

$$P_2(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f(x) = a_0 + a_1 x + a_2 x^2, x \in \mathbb{R} \text{ for some real } a_0, a_1, a_2\}.$$

Show that the linear span of the vectors (or functions) $f_1(x) = 1$, $f_2(x) = 1 + x$, and $f_3(x) = x + x^2$, where $x \in \mathbb{R}$, is P_2 itself.

Solution. We have $f_1 = 1$, $f_2 = 1 + x$, and $f_3 = x + x^2$, $x \in \mathbb{R}$. We have to show that $S_{(f_1,f_2,f_3)} = P_2$. To show equality between two sets, our strategy is to show both that

(i) $S_{(f_1,f_2,f_3)} \subset P_2$, and

(ii) $P_2 \subset S_{(f_1,f_2,f_3)}$.

(i): $S_{(f_1,f_2,f_3)} \subset P_2$: Since each of the functions $f_1(x) = 1$, $f_2(x) = 1 + x$ and $f_3 = x + x^2$ are all in elements of P_2, every linear combination of them is also in P_2, so we have $S_{(f_1,f_2,f_3)} \subset P_2$.

(ii) To show the opposite inclusion $P_2 \subset S_{(f_1,f_2,f_3)}$ we have to show that for any $g(x) = a_0 + a_1x + a_2x^2 \in P_2$ there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha f_1(x) + \beta f_2(x) + \gamma f_3(x) = a_0 + a_1x + a_2x^2$$
$$\text{or } \alpha(1) + \beta(1 + x) + \gamma(x + x^2) = a_0 + a_1x + a_2x^2$$

Re-arranging to get the coefficients of each constant, x and x^2 term:

$$(\alpha + \beta)(1) + (\beta + \gamma)x + \gamma x^2 = a_0(1) + a_1x + a_2x^2$$

When we match the individual coefficients, we can express this as a system of three linear equations in unknowns α, β, γ:

$$\alpha + \beta = a_0 \tag{1}$$
$$\beta + \gamma = a_1 \tag{2}$$
$$\gamma = a_2 \tag{3}$$

We can solve this system by any process you've seen for solving linear equations. Here, we will add and subtract equations.

$$\begin{aligned}(3) \implies \gamma = a_2,\ \text{so } (2) \rightarrow \beta + a_2 &= a_1 \\ \beta &= a_1 - a_2 \qquad (4)\end{aligned}$$

$$\begin{aligned}(4) \rightarrow (1),\ \text{then } \alpha + (a_1 - a_2) &= a_0 \\ \alpha &= a_0 - a_1 + a_2\end{aligned}$$

So, given any $g(x) = a_0 + a_1x + a_2x^2 \in P_2$ we can find scalar multipliers α, β, γ:

$$\begin{aligned}\alpha &= a_0 - a_1 + a_2 \\ \beta &= a_1 - a_2 \\ \text{and } \gamma &= a_2\end{aligned}$$

such that $\alpha(1) + \beta(1 + x) + \gamma(x + x^2) = a_0 + a_1x + a_2x^2 = g(x)$.

This shows that every vector in P_2 is in the span of $f_1 = 1$, $f_2 = 1 + x$, and $f_3 = x + x^2$, or, $P_2 \subset S_{(f_1,f_2,f_3)}$.

With both $P_2 \subset S_{(f_1,f_2,f_3)}$ and $S_{(f_1,f_2,f_3)} \subset P_2$, we have proved that $S_{(f_1,f_2,f_3)} = P_2$.

4. If you compare your solutions to Questions 2 and 3 you will note some similarities. From the perspective of vector spaces (adding and scalar multiplying elements), how can you describe the relationship between $\mathbb{R}^3$ and P_2?

Solution. In Question 2 the vectors we were give were:

$$\mathbf{v}_1 = (1, 0, 0)$$
$$\mathbf{v}_2 = (1, 1, 0)$$
$$\mathbf{v}_3 = (0, 1, 1)$$

In Question 3, if we complete the polynomials with 0 coefficients, the functions we were given were:

$$f_1(x) = 1 + 0x + 0x^2$$
$$f_2(x) = 1 + 1x + 0x^2$$
$$f_3(x) = 0 + 1x + 1x^2$$

Note that the **coefficients** are the same as the earlier **vector components**: $\mathbf{v}_1 = (1,0,0), \mathbf{v}_2 = (1,1,0), \mathbf{v}_3 = (0,1,1)$! As a result, when we looked at the span of those vectors in $\mathbb{R}^3$, and the span of the functions in P_2, we ended up with the same set of equations.

This hints at a **one-to-one relationship between every element in $\mathbb{R}^3$ and P_2**: take the components $(a,b,c) \in \mathbb{R}^3$ and use them as the coefficients of $1, x$, and x^2 respectively for a polynomial in P_2: $a + bx + cx^2$.

If we think of this as a mapping, then vector addition works with the same rules in both $\mathbb{R}^3$ and P_2:

- Adding two vectors is done component by component:
  $(a,b,c) + (d,e,f) = (a+d, b+e, c+f)$.
- Adding two polynomials is done coefficient by coefficient:
  $(a + bx + cx^2) + (d + ex + fx^2) = (a+d) + (b+e)x + (c+f)x^2$.

Scalar multiplication also works with the same rules for both $\mathbb{R}^3$ and P_2:

- Multiplying a vector by a scalar is done by distributing over each component:
  $\alpha(a,b,c) = (\alpha a,\, \alpha b,\, \alpha c)$.
- Multiplying a polynomial by a real value is done distributing over each coefficient:
  $\alpha(a + bx + cx^2) = (\alpha a) + (\alpha b)x + (\alpha c)x^2$.

So at least from a vector space perspective, any statements or proofs we can make using elements of $\mathbb{R}^3$ will lead directly to analogous statements and proofs for quadratic functions in P_2.

5. Let $\mathbf{V}$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in \mathbf{V}$ $(n \geqslant 2)$. Show that if $S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_n)} = S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_{n-1})}$, then $\mathbf{v}_n \in S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_{n-1})}$.

Solution. The problem asks us to show that if the span of the n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the same as the span of the first $n-1$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$, then $\mathbf{v}_n$ must be in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$.

Recall that $S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_n)}$ is the set of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. But setting $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$ and $\alpha_n = 1$, we can write $\mathbf{v}_n$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\begin{aligned} \mathbf{v}_n &= 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \cdots + 0 \cdot \mathbf{v}_{n-1} + 1 \cdot \mathbf{v}_n \\ &= \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-1}\mathbf{v}_{n-1} + \alpha_n \mathbf{v}_n. \end{aligned}$$

Thus $\mathbf{v}_n \in S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_n)}$ and since $S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_n)} = S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_{n-1})}$ by assumption, we obtain $\mathbf{v}_n \in S_{(\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_{n-1})}$, which is what we had to prove.

6. In class we have seen that if $\mathbf{v}_1, \ldots, \mathbf{v}_p$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then we can convert the question "Is $\mathbf{w}$ a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$?" into a question "Does this system of linear equations have a solution?".

(a) Let $\mathbf{w} = (1,3,5,3)$, $\mathbf{v}_1 = (2,4,3,0)$, $\mathbf{v}_2 = (1,1,-2,-1)$, $\mathbf{v}_3 = (0,2,7,3)$. What system of linear equations does the question "Is $\mathbf{w}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?" correspond to?

(Note: You do not have to solve the system of linear equations — the purpose of this question is just to write them down.)

(b) Conversely, consider the system of linear equations below :

$$\begin{array}{rcrcrcrcl} 2x_1 & - & x_2 & + & 4x_3 & + & 9x_4 & = & 0 \\ x_1 & + & 5x_2 & + & x_3 & - & 4x_4 & = & 8 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 1. \end{array}$$

Which "linear combination" problem does this system correspond to?

Solution.

(a) The equation representing the question "is $\mathbf{w}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?" is

$$x_1 \cdot \mathbf{v}_1 + x_2 \cdot \mathbf{v}_2 + x_3 \cdot \mathbf{v}_3 = \mathbf{w}.$$

Writing the vectors as column vectors this is

$$x_1 \begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ -2 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 2 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}$$

or

$$\begin{pmatrix} 2x_1 + x_2 + 0x_3 \\ 4x_1 + x_2 + 2x_3 \\ 3x_1 - 2x_2 + 7x_3 \\ 0x_1 - x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}.$$

Comparing entries of the vectors we get the following system of linear equations :

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & & & = & 1 \\ 4x_1 & + & x_2 & + & 2x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 7x_2 & = & 5 \\ & - & x_2 & + & 3x_3 & = & 3. \end{array}$$

(b) The fastest way to see the answer is to write the linear system of equations as a single vector equation with column vectors :

$$x_1 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + x_4 \begin{pmatrix} 9 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 1 \end{pmatrix}.$$

From this we see that if we set $\mathbf{v}_1 = (2, 1, 1)$, $\mathbf{v}_2 = (-1, 5, 0)$, $\mathbf{v}_3 = (4, 1, 2)$, $\mathbf{v}_4 = (9, -4, 4)$ and $\mathbf{w} = (0, 8, 1)$, the question "Does the system of linear equations above have a solution?" is the same as the question "Is $\mathbf{w}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$?".

7. Let

$$\mathbf{W} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ 3x - 2y + z = 0 \right\}.$$

(a) Show, with proof, that $\mathbf{W}$ is a subspace of $\mathbb{R}^3$.

(b) Let $\mathbf{v}_1 = (1, 0, -3)$ and $\mathbf{v}_2 = (0, 1, 2)$. Check that $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{W}$.

(c) Show that $S_{(\mathbf{v}_1, \mathbf{v}_2)} \subset \mathbf{W}$.

(d) Check that $(4, 5, -2) \in \mathbf{W}$.

(e) Write $(4, 5, -2)$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(f) Show that $\mathbf{W} \subset S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

(g) Show that $\mathbf{W} = S_{(\mathbf{v}_1, \mathbf{v}_2)}$.

Solution.

(a) We check the three conditions to be a subspace.

(I) **Q:** Is $\mathbf{0} \in \mathbf{W}$?

A: Yes. In $\mathbb{R}^3$, $\mathbf{0} = (0, 0, 0)$ and since $3 \cdot 0 - 2 \cdot 0 + 0 = 0$, we conclude that $\mathbf{0} \in \mathbf{W}$.

(II) **Q:** If $\mathbf{w}_1,\ \mathbf{w}_2 \in \mathbf{W}$, is $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}$?

A: Yes. Suppose that $\mathbf{w}_1 = (x_1, y_1, z_1)$. Since $\mathbf{w}_1 \in \mathbf{W}$ we know that $3x_1 - 2y_1 + z_1 = 0$. Similarly, let $\mathbf{w}_2 = (x_2, y_2, z_2)$. Since $\mathbf{w}_2 \in \mathbf{W}$ we know that $3x_2 - 2y_2 + z_2 = 0$.

Then $\mathbf{w}_1 + \mathbf{w}_2 = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$, and this vector is in $\mathbf{W}$ since

$$\begin{aligned} 3(x_1+x_2) - 2(y_1+y_2) + (z_1+z_2) &= 3x_1 + 3x_2 - 2y_1 - 2y_2 + z_1 + z_2 \\ &= (3x_1 - 2y_1 + z_1) + (3x_2 - 2y_2 + z_2) \\ &= 0 + 0 = 0. \end{aligned}$$

(III) If $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbf{W}$, is $\alpha \cdot \mathbf{w} \in \mathbf{W}$?

A: Yes. Let $\mathbf{w} = (x, y, z)$. Since $\mathbf{w} \in \mathbf{W}$ we know that $3x - 2y + z = 0$. Then $\alpha \cdot \mathbf{w} = (\alpha x,\ \alpha y,\ \alpha z)$, and $\alpha \cdot \mathbf{w} \in \mathbf{W}$ since

$$3(\alpha x) - 2(\alpha y) + (\alpha z) = \alpha(3x - 2y + z) = \alpha \cdot 0 = 0.$$

Since $\mathbf{W}$ passes all three tests, $\mathbf{W}$ is a subspace of $\mathbb{R}^3$.

(b) Since $3 \cdot 1 - 2 \cdot 0 + 1 \cdot (-3) = 0$, the vector $\mathbf{v}_1 = (1, 0, -3)$ is in $\mathbf{W}$. Similarly, since $3 \cdot 0 - 2 \cdot 1 + 1 \cdot 2 = 0$, $\mathbf{v}_2 = (0, 1, 2) \in \mathbf{W}$.

(c) <u>Argument 1:</u> In part (b) we checked that $\mathbf{v}_1$ and $\mathbf{v}_2 \in \mathbf{W}$. We know that $\mathbf{W}$ is a subspace of $\mathbb{R}^3$ by (a). Using condition (*iii*) to be a subspace we conclude that for any $\alpha_1,\ \alpha_2 \in \mathbb{R}$, $\alpha_1 \cdot \mathbf{v}_1$ and $\alpha_2 \cdot \mathbf{v}_2$ are both in $\mathbf{W}$. Using condition (*ii*) to be a subspace we then conclude that $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 \in \mathbf{W}$.

Therefore, all vectors of the form $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2$ are in $\mathbf{W}$. The set of vectors of this form is exactly $S_{(\mathbf{v}_1,\mathbf{v}_2)}$, and so $S_{(\mathbf{v}_1,\mathbf{v}_2)} \subset \mathbf{W}$.

<u>Argument 2:</u> The vectors in $S_{(\mathbf{v}_1,\mathbf{v}_2)}$ are all the vectors of the form $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2$ with $\alpha_1,\ \alpha_2 \in \mathbb{R}$. Expanding, we have

$$\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 = \alpha_1(1, 0, -3) + \alpha_2(0, 1, 2) = (\alpha_1,\ \alpha_2, -3\alpha_1 + 2\alpha_2).$$

Applying the test to be in $\mathbf{W}$ to this vector we get

$$3\alpha_1 - 2\alpha_2 + (-3\alpha_1 + 2\alpha_2) = (3\alpha_1 - 3\alpha_1) + (-2\alpha_2 + 2\alpha_2) = 0.$$

Therefore no matter what α_1 and α_2 are, $\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 \in \mathbf{W}$, and so $S_{(\mathbf{v}_1,\mathbf{v}_2)} \subset \mathbf{W}$.

(d) Since $3 \cdot 4 - 2 \cdot 5 + 1 \cdot (-2) = 0$, $(4, 5, -2) \in \mathbf{W}$.

(e) We are trying to solve

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 = \alpha_1(1, 0, -3) + \alpha_2(0, 1, 2) = (4, 5, -2),$$

which gives the system

$$\begin{array}{rcrcr} \alpha_1 & & & = & 4 \\ & & \alpha_2 & = & 5 \\ -3\alpha_1 & + & 2\alpha_2 & = & -2. \end{array}$$

The first two equations give us $\alpha_1 = 4$ and $\alpha_2 = 5$. Since this is also a solution of the third equation, $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(f) Suppose that $\mathbf{w} = (x, y, z) \in \mathbf{W}$. We want to show that $\mathbf{w} \in S_{(\mathbf{v}_1,\mathbf{v}_2)}$, so we want to show that $\mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ for some $\alpha_1,\ \alpha_2 \in \mathbb{R}$. As in (e) this gives us the system of equations

$$\begin{array}{rcrcl} \alpha_1 & & & = & x \\ & & \alpha_2 & = & y \\ -3\alpha_1 & + & 2\alpha_2 & = & z. \end{array}$$

From the first two equations we must have $\alpha_1 = x$ and $\alpha_2 = y$. Is this also a solution to the third equation?

Since $\mathbf{w} \in \mathbf{W}$ we know that $3x - 2y + z = 0$, or $z = -3x + 2y$, and so setting $\alpha_1 = x$ and $\alpha_2 = y$, we do have $-3\alpha_1 + 2\alpha_2 = -3x + 2y = z$. Therefore $\alpha_1 = x$ and $\alpha_2 = y$ is a solution to all three equations, and so $\mathbf{w}$ is in $S_{(\mathbf{v}_1,\mathbf{v}_2)}$.

We have just shown that for any $\mathbf{w} \in \mathbf{W}$ we also have $\mathbf{w} \in S_{(\mathbf{v}_1,\mathbf{v}_2)}$. We conclude that $\mathbf{W} \subset S_{(\mathbf{v}_1,\mathbf{v}_2)}$.

(g) From (c) we have $S_{(\mathbf{v}_1,\mathbf{v}_2)} \subset \mathbf{W}$, and from (f) that $\mathbf{W} \subset S_{(\mathbf{v}_1,\mathbf{v}_2)}$. Therefore $\mathbf{W} = S_{(\mathbf{v}_1,\mathbf{v}_2)}$.

8. For each of the following sets of vectors, determine if they are linearly dependent or linearly independent.

(a) $\{(1,2),\ (5,1),(1,0)\} \subset \mathbb{R}^2$.

(b) $\{(2,1,3),(-1,2,1),\ (0,4,5)\} \subset \mathbb{R}^3$.

(c) $\{(2,0,3),(-1,2,1),\ (0,4,5)\} \subset \mathbb{R}^3$.

(d) $\{(1,2,4),\ (0,0,0),\ (2,3,1)\} \subset \mathbb{R}^3$.

(e) $\{\sin(x),\ \cos(x),\ 1\} \subset C^\infty(\mathbb{R})$.

(f) $\{\sin^2(x),\ \cos^2(x),\ 1\} \subset C^\infty(\mathbb{R})$.

(g) $\{1,\ x,\ x^2\} \subset C^\infty(\mathbb{R})$.

(h) $\{(2,3),\ (3,5)\} \subset \mathbf{W}_2$

(i) $\{(2,3),\ (1,1)\} \subset \mathbf{W}_2$

Notes:

(1) When we write something like "1"for an element of $C^\infty(\mathbb{R})$ we mean the constant function 1. That is, the function $f(x)$ defined by $f(x) = 1$ for all $x \in \mathbb{R}$. Similarly for any other number.

(2) When trying to decide of a set of functions is linearly dependent or linearly independent, as in (e)–(g), one way to go from an equation like $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$ to equations with numbers is to plug in different values of x.

(3) In parts (h) and (i) of question 8 the space $\mathbf{W}_2$ is the "weird" vector space introduced in class. Namely, as a set $\mathbf{W}_2 = \{(x,y)\ :\ x,y \in \mathbb{R},\ x,y > 0\}$ with operations $(x_1,y_1)+(x_2,y_2) = (x_1x_2, y_1,y_2)$ and $\alpha\cdot(x,y) = (x^\alpha, y^\alpha)$.

Solution. To determine whether a set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots, \mathbf{v}_p\}$ of vectors is linearly dependent or independent, we write down the equation of possible linear dependence :

$$\alpha_1 \cdot \mathbf{v}_1 + \alpha_2 \cdot \mathbf{v}_2 + \cdots + \alpha_p \cdot \mathbf{v}_p = \mathbf{0}$$

and look for solutions.

If the <u>only</u> solution to this equation is $\alpha_1 = 0,\ \alpha_2 = 0, \ldots,\ \alpha_p = 0$, then the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots, \mathbf{v}_p\}$ is *linearly independent*. If there is a solution <u>other than</u> $\alpha_1 = 0,\ \alpha_2 = 0, \ldots,\ \alpha_p = 0$, then the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots, \mathbf{v}_p\}$ is *linearly dependent*.

We now do this in each of the questions.

(a) The equation is $\alpha_1(1,2) + \alpha_2(5,1) + \alpha_3(1,0) = \mathbf{0} = (0,0)$, equivalent to the system

$$\begin{array}{rcrcrcl} \alpha_1 & + & 5\alpha_2 & + & \alpha_3 & = & 0 \\ 2\alpha_1 & + & \alpha_2 & & & = & 0 \end{array}$$

The second equation tells us that we need to have $\alpha_2 = -2\alpha_1$. Substituting this into the first equation gives

$$0 = \alpha_1 + 5(-2\alpha_1) + \alpha_3 = -9\alpha_1 + \alpha_3,$$

or $\alpha_3 = 9\alpha_1$. We now have α_2 and α_3 written in terms of α_1, and there are no other conditions. Setting (for example) $\alpha_1 = 1$, this gives $\alpha_2 = -2 \cdot 1 = -2$ and $\alpha_3 = 9 \cdot 1 = 9$.

Since these values do solve the system :

$$\begin{array}{rcrcrl} 1 & + & 5(-2) & + & 9 = & 0 \\ 2(1) & + & (-2) & & = & 0, \end{array}$$

and since at least one (in this case all) of the α's is not zero, the set $\{(1,2),\ (5,1),\ (1,0)\}$ is *linearly dependent*.

(b) The equation is $\alpha_1(2,1,3)+\alpha_2(-1,2,1)+\alpha_3(0,4,5)=\mathbf{0}=(0,0,0)$, equivalent to the system

$$\begin{array}{rcrcrcl} 2\alpha_1 & - & \alpha_2 & & & = & 0 \\ \alpha_1 & + & 2\alpha_2 & + & 4\alpha_3 & = & 0 \\ 3\alpha_1 & + & \alpha_2 & + & 5\alpha_3 & = & 0. \end{array}$$

From the first equation we get $\alpha_2 = 2\alpha_1$. Substituting this identity into the second and third equations they become

$$\begin{array}{rcrcl} 5\alpha_1 & + & 4\alpha_3 & = & 0 \\ 5\alpha_1 & + & 5\alpha_3 & = & 0. \end{array}$$

This last equation is the same as $\alpha_1+\alpha_3=0$, so $\alpha_3=-\alpha_1$. Substituting this into the equation $5\alpha_1+4\alpha_3=0$, we get $5\alpha_1+4(-\alpha_1)=\alpha_1=0$. Therefore $\alpha_1=0$. Finally, we already know that $\alpha_2=2\alpha_1$ and $\alpha_3=-\alpha_1$, so α_2 and α_3 are both zero too.

Starting only with the assumption that α_1, α_2, and α_3 were solutions to the original system of linear equations, we have deduced that all the α's must be zero. Thus these calculations show that the <u>only</u> solution to the original system of linear equations is $\alpha_1=0$, $\alpha_2=0$, and $\alpha_3=0$, and therefore the set $\{(2,1,3),\ (-1,2,1),\ (0,4,5)\}$ is *linearly independent*.

(c) The equation is $\alpha_1(2,0,3)+\alpha_2(-1,2,1)+\alpha_3(0,4,5)=\mathbf{0}=(0,0,0)$, equivalent to the system

$$\begin{array}{rcrcrcl} 2\alpha_1 & - & \alpha_2 & & & = & 0 \\ & & 2\alpha_2 & + & 4\alpha_3 & = & 0 \\ 3\alpha_1 & + & \alpha_2 & + & 5\alpha_3 & = & 0. \end{array}$$

As before, the first equation gives us $\alpha_2=2\alpha_1$. Substituting this identity into the second and third equations they become

$$\begin{array}{rcrcl} 4\alpha_1 & + & 4\alpha_3 & = & 0 \\ 5\alpha_1 & + & 5\alpha_3 & = & 0. \end{array}$$

But these are really both the same condition, that $\alpha_1+\alpha_3=0$, so $\alpha_3=-\alpha_1$. So, the conditions we need to satisfy are $\alpha_2=2\alpha_1$ and $\alpha_3=-\alpha_1$. Let us just pick a value of α_1, say $\alpha_1=1$. Then these conditions give $\alpha_2=2\cdot 1=2$, and $\alpha_3=-1$. This is a solution of the system of linear equations. We can verify by substituting them back into the original system of linear equations, or by checking the linear combination to see what happens. With these values,

$$\begin{aligned} \alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\alpha_3\mathbf{v}_3 &= 1(2,0,3)+2(-1,2,1)-(0,4,5) \\ &= \big(1\cdot 2+2(-1)-1\cdot 0,\ 1\cdot 0+2\cdot 2-4,\ 1\cdot 3+2\cdot 1-5\big) \\ &= (0,0,0)=\mathbf{0}. \end{aligned}$$

Since the coefficients α_1, α_2, and α_3 are not all zero, the set $\{(2,0,3),\ (-1,2,1),\ (0,4,5)\}$ is *linearly dependent*.

(d) The equation is $\alpha_1(1,2,4)+\alpha_2(0,0,0)+\alpha_3(2,3,1)=\mathbf{0}=(0,0,0)$. Since the second vector is the zero vector, there is a simple non-zero solution to the vector equation. Set $\alpha_1=0$, $\alpha_2=1$, and $\alpha_3=0$. Then

$$0\cdot(1,2,4)+1\cdot(0,0,0)+0\cdot(2,3,1)=(0,0,0).$$

Since $\alpha_2\neq 0$ (so this is a solution different from $\alpha_1=0$, $\alpha_2=0$, and $\alpha_3=0$), the set $\{(1,2,4),\ (0,0,0),\ (2,3,1)\}$ is *linearly dependent*.

NOTE: This reasoning works when any of the vectors $\mathbf{v}_1,\ \ldots,\ \mathbf{v}_p$ is the zero vector, and thus, in such a case, $\{\mathbf{v}_1,\ \mathbf{v}_2,\ldots,\mathbf{v}_p\}$ is always linearly dependent.

(e) The equation is

$$\alpha_1\cdot\sin(x)+\alpha_2\cdot\cos(x)+\alpha_3\cdot 1=\mathbf{0}(x)=0$$

for all $x\in\mathbb{R}$. Here, $\mathbf{0}$ means the *zero function*, the function which, for every input x outputs the number 0. Similarly, 1 means the *constant function* 1, the function which, for every input x, outputs the number 1.

In parts (a)–(d) we converted the vector equation in $\mathbb{R}^2$ or $\mathbb{R}^3$ into a system of linear equations in order to look for solutions. One way to get a system of equations out of the single equation above is to substitute in different values of x. For instance,

- Plugging in $x = 0$ gives

$$0 = \alpha_1 \sin(0) + \alpha_2 \cos(0) + \alpha_3 \cdot 1 = \alpha_1 \cdot 0 + \alpha_2 \cdot 1 + \alpha_3 = \alpha_2 + \alpha_3.$$

- Plugging in $x = \frac{\pi}{2}$ gives

$$0 = \alpha_1 \sin\left(\tfrac{\pi}{2}\right) + \alpha_2 \cos\left(\tfrac{\pi}{2}\right) + \alpha_3 \cdot 1 = \alpha_1 \cdot 1 + \alpha_2 \cdot 0 + \alpha_3 = \alpha_1 + \alpha_3.$$

- Plugging in $x = \frac{\pi}{4}$ gives

$$0 = \alpha_1 \sin\left(\tfrac{\pi}{4}\right) + \alpha_2 \cos\left(\tfrac{\pi}{4}\right) + \alpha_3 \cdot 1 = \alpha_1 \cdot \tfrac{1}{\sqrt{2}} + \alpha_2 \cdot \tfrac{1}{\sqrt{2}} + \alpha_3 = \tfrac{\alpha_1}{\sqrt{2}} + \tfrac{\alpha_2}{\sqrt{2}} + \alpha_3.$$

This gives us the system of linear equations

$$\begin{array}{ccccccc} & & \alpha_2 & + & \alpha_3 & = & 0 \\ \alpha_1 & & & + & \alpha_3 & = & 0 \\ \frac{1}{\sqrt{2}}\alpha_1 & + & \frac{1}{\sqrt{2}}\alpha_2 & + & \alpha_3 & = & 0. \end{array}$$

If α_1, α_2, and α_3 are numbers so that the original equation

$$\alpha_1 \cdot \sin(x) + \alpha_2 \cos(x) + \alpha_3 \cdot 1 = 0$$

holds, then α_1, α_2, and α_3 must also be solutions to the system above. So, we can deduce information about the α's by studying that system.

The first two equations say that $\alpha_2 = -\alpha_3$ and $\alpha_1 = -\alpha_3$. Substituting this into the third equation we get

$$0 = \tfrac{1}{\sqrt{2}}(-\alpha_3) + \tfrac{1}{\sqrt{2}}(-\alpha_3) + \alpha_3 = \left(-\tfrac{2}{\sqrt{2}} + 1\right)\alpha_3 = \left(1 - \sqrt{2}\right)\alpha_3,$$

and dividing by $(1 - \sqrt{2})$, that $\alpha_3 = 0$. Therefore $\alpha_1 = -\alpha_3 = -0 = 0$ and $\alpha_2 = -\alpha_3 = -0 = 0$ too.

Since $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$ is the only solution to the system of linear equations, the set $\{\sin(x), \cos(x), 1\}$ is *linearly independent*.

(f) The equation is

$$\alpha_1 \cdot \sin^2(x) + \alpha_2 \cdot \cos^2(x) + \alpha_3 \cdot 1 = \mathbf{0}(x) = 0$$

for all $x \in \mathbb{R}$. In this case we already know a solution to the equations. If $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_3 = -1$ then we have

$$1 \cdot \sin^2(x) + 1 \cdot \cos^2(x) + (-1) \cdot 1 = \sin^2(x) + \cos^2(x) - 1 = 0.$$

Since this is a non-zero solution to the equation, the set $\{\sin^2(x), \cos^2(x), 1\}$ is *linearly dependent*.

(g) The equation is

$$\alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2 = \mathbf{0}(x) = 0$$

for all $x \in \mathbb{R}$. Here, as above, "$\mathbf{0}$" means the zero vector of $C^\infty(\mathbb{R})$, that is the zero function. For the equation above to hold, it means that for every $x \in \mathbb{R}$, if we plug x into the function $\alpha_1 + \alpha_2 x + \alpha_3 x^2$, the result is zero.

For instance,

- Picking $x = 0$, it would mean that
$$\alpha_1 + \alpha_2 \cdot 0 + \alpha_3 \cdot 0^2 = 0.$$
- Picking $x = 1$, it would mean that
$$\alpha_1 + \alpha_2 \cdot 1 + \alpha_3 \cdot 1^2 = 0.$$
- Picking $x = -1$, it would mean that
$$\alpha_1 + \alpha_2 \cdot (-1) + \alpha_3 \cdot (-1)^2 = 0.$$

Then we have the following system of linear equations:

$$\begin{array}{ccccccc} \alpha_1 & & & & & = & 0 \\ \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & 0 \\ \alpha_1 & - & \alpha_2 & + & \alpha_3 & = & 0. \end{array}$$

The first equation already shows that $\alpha_1 = 0$. Substituting this into the remaining two equations we get

$$\begin{array}{ccccc} \alpha_2 & + & \alpha_3 & = & 0 \\ -\alpha_2 & + & \alpha_3 & = & 0. \end{array}$$

Adding the two equations together gives $2\alpha_3 = 0$, so $\alpha_3 = 0$, and substituting this into either of the equations gives $\alpha_2 = 0$. That is, the only solution to the three equations is $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. Therefore $\{1,\, x,\, x^2\}$ is linearly independent.

(h) The equation is
$$\alpha_1 \cdot (2,3) + \alpha_2 \cdot (3,5) = \mathbf{0} = (1,1).$$

Using the rules in $\mathbf{W}_2$, we compute that

$$\alpha_1 \cdot (2,3) + \alpha_2 \cdot (3,5) = (2^{\alpha_1},\, 3^{\alpha_1}) + (3^{\alpha_2},\, 5^{\alpha_2}) = (2^{\alpha_1}3^{\alpha_2},\, 3^{\alpha_1}5^{\alpha_2}),$$

and so the equation is equivalent to the system

$$\begin{array}{ccc} 2^{\alpha_1}3^{\alpha_2} & = & 1 \\ 3^{\alpha_1}5^{\alpha_2} & = & 1. \end{array}$$

This is not a system of linear equations, but we can turn it into one by taking the logarithm :

$$\begin{array}{cccccc} \ln(2)\alpha_1 & + & \ln(3)\alpha_2 & = & 0 & (= \ln(1)) \\ \ln(3)\alpha_1 & + & \ln(5)\alpha_2 & = & 0 & (= \ln(1)). \end{array}$$

Multiplying the top equation by $\ln(3)$, the bottom equation by $\ln(2)$ and subtracting, we get

$$\left(\ln(3)^2 - \ln(5)\ln(2)\right)\alpha_2 = 0.$$

Since $\ln(3)^2 - \ln(5)\ln(2) \neq 0$, this means we must have $\alpha_2 = 0$. Substituting into either of the first two equations we conclude that $\alpha_1 = 0$ too.

Therefore the only solution is $\alpha_1 = 0$ and $\alpha_2 = 0$, and so $\{(2,3),\, (3,5)\}$ are *linearly independent* in $\mathbf{W}_2$.

(i) The set $\{(2,3),\, (1,1)\}$ is *linearly dependent*, for the same reason as in part (d): the zero vector is in the set. So, taking α_2 to be any nonzero number (for instance, $\alpha_2 = 5$) and $\alpha_1 = 0$ we get

$$0 \cdot (2,3) + 5 \cdot (1,1) = (2^0,\, 3^0) + (1^5,\, 1^5) = (1,1) + (1,1) = (1,1) = \mathbf{0} \in \mathbf{W}_2.$$

Note: The operations $+$ and $\cdot$ in part (h) and (i) are the operations from the vector space $(\mathbf{W}_2, +, \cdot)$.