Queen's University - APSC174 **Winter 2025**

Tutorial 05

1. Consider the vector space $\mathbb{R}^2$ with the usual vector addition and scalar multiplication operations. For $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, 4)$, let $S_{(\mathbf{u})}$ and $S_{(\mathbf{v})}$ be the span of $\{\mathbf{u}\}$ and $\{\mathbf{v}\}$, respectively. Find $\mathbf{w}_1 \in S_{(\mathbf{u})}$ and $\mathbf{w}_2 \in S_{(\mathbf{v})}$ such that

$$\mathbf{w}_1 + \mathbf{w}_2 = (2, 3).$$

2. Let $\mathbf{V}$ be a vector space and let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$. Define $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$. Prove that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.

3. Let $(\mathbf{V}, +, \cdot)$ be a real vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{V}$. Let $\mathbf{W}_1$ denote the linear span of the vectors $\mathbf{v}_1, \mathbf{v}_2$, and let $\mathbf{W}_2$ denote the linear span of the vectors $\mathbf{v}_3, \mathbf{v}_4$. Assume $\mathbf{v}_3 \in \mathbf{W}_1$ and $\mathbf{v}_4 \in \mathbf{W}_1$; show that it then follows that $\mathbf{W}_2 \subset \mathbf{W}_1$.

4. Let $(\mathbf{V}, +, \cdot)$ be a real vector space, and let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ with $\mathbf{v}_1 \neq \mathbf{v}_2$. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{V}$ be defined by $\mathbf{w}_1 = 2\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - 2\mathbf{v}_2$. Show that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent as well.

5. In the vector space $\mathbb{R}^2$ consider the vectors $(6, 2)$ and $(4, t)$, where $t \in \mathbb{R}$ is a real parameter.

(a) Find *all the values* of t such that the vector equation

$$x_1(6, 2) + x_2(4, t) = (1, 1)$$

has a solution.

(b) Find *all the values* of t such that the vector equation

$$x_1(6, 2) + x_2(4, t) = \mathbf{b}$$

always has a solution regardless of the choice of $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$.

(c) For which value(s) of t does the vector equation

$$x_1(6, 2) + x_2(4, t) = (9, 3)$$

have an *infinite number* of solutions?