1. Consider the vector space $\mathbb{R}^2$ with the usual vector addition and scalar multiplication operations. For $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, 4)$, let $S_{(\mathbf{u})}$ and $S_{(\mathbf{v})}$ be the span of $\{\mathbf{u}\}$ and $\{\mathbf{v}\}$, respectively. Find $\mathbf{w}_1 \in S_{(\mathbf{u})}$ and $\mathbf{w}_2 \in S_{(\mathbf{v})}$ such that

$$\mathbf{w}_1 + \mathbf{w}_2 = (2, 3).$$

Solution. In general, the span of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ consists of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Here $\{\mathbf{u}\}$ and $\{\mathbf{v}\}$ contain only one vector each, so $S_{(\mathbf{u})}$ is the set of all vectors in the form $\alpha\mathbf{u}$, $\alpha \in \mathbb{R}$, and $S_{(\mathbf{v})}$ is the set of all vectors in the form $\beta\mathbf{v}$, $\beta \in \mathbb{R}$. Thus we are looking for $\mathbf{w}_1 = \alpha\mathbf{u} = \alpha(1, 1)$ and $\mathbf{w}_2 = \beta\mathbf{v} = \beta(1, 4)$ such that

$$\mathbf{w}_1 + \mathbf{w}_2 = (2, 3),$$

i.e., we need to find $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha(1, 1) + \beta(1, 4) = (2, 3).$$

This vector equation is equivalent to the system of linar equations

$$\begin{aligned} \alpha + \beta &= 2 \\ \alpha + 4\beta &= 3. \end{aligned}$$

The unique solution is $\alpha = 5/3$ and $\beta = 1/3$ and so the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are given by

$$\mathbf{w}_1 = \frac{5}{3}(1, 1) = \left(\frac{5}{3}, \frac{5}{3}\right), \qquad \mathbf{w}_2 = \frac{1}{3}(1, 4) = \left(\frac{1}{3}, \frac{4}{3}\right).$$

2. Let $\mathbf{V}$ be a vector space and let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$. Define $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$. Prove that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.

Solution. By definition, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if the only solution to the vector equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \mathbf{0} \tag{1}$$

is $\alpha_1 = \alpha_2 = 0$. Similarly, $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent if the only solution to the vector equation

$$\beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 = \mathbf{0} \tag{2}$$

is $\beta_1 = \beta_2 = 0$.

We have to prove that the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2\}$ implies that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent and also that the linear independence of $\{\mathbf{w}_1, \mathbf{w}_2\}$ implies that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

First, we will prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent. To do this, substitute $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$ into equation (2) to obtain

$$\beta_1(\mathbf{v}_1 + \mathbf{v}_2) + \beta_2(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0},$$

i.e., after rearranging terms,

$$(\beta_1 + \beta_2)\mathbf{v}_1 + (\beta_1 - \beta_2)\mathbf{v}_2 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, this last vector equation holds exactly when $\beta_1 + \beta_2 = 0$ and $\beta_1 - \beta_2 = 0$. From the second equation we get $\beta_1 = \beta_2$, which, upon substitution into the first equation, gives $2\beta_1 = 0$, i.e., $\beta_1 = 0$, which then implies $\beta_2 = 0$.

Thus we have proved that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then the vector equation (2) has the unique solution $\beta_1 = \beta_2 = 0$, implying that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent.

Conversely, we now assume that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent and prove that this implies that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. To do this, we will express $\mathbf{v}_1$ and $\mathbf{v}_2$ in terms of $\mathbf{w}_1$ and $\mathbf{w}_2$. First notice that adding the vector equations $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2$ gives $\mathbf{w}_1 + \mathbf{w}_2 = 2\mathbf{v}_1$, i.e.,

$$\mathbf{v}_1 = \frac{1}{2}\big(\mathbf{w}_1 + \mathbf{w}_2\big).$$

Substituting this into $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ in turn gives $\mathbf{w}_1 = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2 + \mathbf{v}_2$, i.e.,

$$\mathbf{v}_2 = \frac{1}{2}\big(\mathbf{w}_1 - \mathbf{w}_2\big).$$

Now we can use the assumed linear independence of $\{\mathbf{w}_1, \mathbf{w}_2\}$ by substituting these two equalities into equation (1). We obtain

$$\alpha_1 \cdot \underbrace{\frac{1}{2}\big(\mathbf{w}_1 + \mathbf{w}_2\big)}_{\mathbf{v}_1} + \alpha_2 \cdot \underbrace{\frac{1}{2}\big(\mathbf{w}_1 - \mathbf{w}_2\big)}_{\mathbf{v}_2} = \mathbf{0},$$

i.e.,

$$\frac{1}{2}(\alpha_1 + \alpha_2)\mathbf{w}_1 + \frac{1}{2}(\alpha_1 - \alpha_2)\mathbf{w}_2 = \mathbf{0}.$$

Since $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent, this last vector equation holds if and only if $\frac{1}{2}(\alpha_1 + \alpha_2) = 0$ and $\frac{1}{2}(\alpha_1 - \alpha_2) = 0$. From the second equation we have $\alpha_1 = \alpha_2$. Substituting this into the first equation gives $\alpha_1 = 0$, which in turn gives $\alpha_2 = 0$.

Thus if $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent, then the vector equation (1) has the unique solution $\alpha_1 = \alpha_2 = 0$, implying that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

3. Let $(\mathbf{V}, +, \cdot)$ be a real vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{V}$. Let $\mathbf{W}_1$ denote the linear span of the vectors $\mathbf{v}_1, \mathbf{v}_2$, and let $\mathbf{W}_2$ denote the linear span of the vectors $\mathbf{v}_3, \mathbf{v}_4$. Assume $\mathbf{v}_3 \in \mathbf{W}_1$ and $\mathbf{v}_4 \in \mathbf{W}_1$; show that it then follows that $\mathbf{W}_2 \subset \mathbf{W}_1$.

Solution. We assume $\mathbf{v}_3, \mathbf{v}_4 \in \mathbf{W}_1$. By definition of $\mathbf{W}_1$, this means $\mathbf{v}_3$ and $\mathbf{v}_4$ can both be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. In other words, there exist $a, b \in \mathbb{R}$ such that

$$\mathbf{v}_3 = a \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2,$$

and there exist $c, d \in \mathbf{R}$ such that

$$\mathbf{v}_4 = c \cdot \mathbf{v}_1 + d \cdot \mathbf{v}_2.$$

Let now $\mathbf{w}$ be an arbitrary element of $\mathbf{W}_2$. By definition of $\mathbf{W}_2$, this means $\mathbf{w}$ is a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_4$, that is, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{w} = \alpha \cdot \mathbf{v}_3 + \beta \cdot \mathbf{v}_4,$$

which yields

$$\mathbf{w} = \alpha \cdot (a \cdot \mathbf{v}_1 + b \cdot \mathbf{v}_2) + \beta \cdot (c \cdot \mathbf{v}_1 + d \cdot \mathbf{v}_2),$$

which can be rewritten as

$$\mathbf{w} = (\alpha a + \beta c) \cdot \mathbf{v}_1 + (\alpha b + \beta d) \cdot \mathbf{v}_2,$$

which shows that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and hence an element of $\mathbf{W}_1$. We have thus shown that any element of $\mathbf{W}_2$ is also an element of $\mathbf{W}_1$, that is, $\mathbf{W}_2 \subset \mathbf{W}_1$.

4. Let $(\mathbf{V}, +, \cdot)$ be a real vector space, and let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$ with $\mathbf{v}_1 \neq \mathbf{v}_2$. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{V}$ be defined by $\mathbf{w}_1 = 2\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - 2\mathbf{v}_2$. Show that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent as well.

Solution. Assume now that the subset $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. We wish to show that the subset $\{\mathbf{w}_1, \mathbf{w}_2\}$ is then linearly independent as well. Let then $a, b \in \mathbb{R}$ such that

$$a \cdot \mathbf{w}_1 + b \cdot \mathbf{w}_2 = \mathbf{0},$$

i.e. such that

$$a \cdot (2\mathbf{v}_1 + \mathbf{v}_2) + b \cdot (\mathbf{v}_1 - 2\mathbf{v}_2) = \mathbf{0},$$

which yields

$$(2a + b) \cdot \mathbf{v}_1 + (a - 2b) \cdot \mathbf{v}_2 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is assumed linearly independent, it follows that

$$2a + b = 0 \quad \text{and} \quad a - 2b = 0,$$

which then yields $a = b = 0$.

We have therefore shown that the only allowed values of a and b so that

$$a \cdot \mathbf{w}_1 + b \cdot \mathbf{w}_2 = \mathbf{0}$$

holds are $a = 0$ and $b = 0$. Hence, we can conclude that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a linearly independent subset of $(\mathbf{V}, +, \cdot)$.

5. In the vector space $\mathbb{R}^2$ consider the vectors $(6, 2)$ and $(4, t)$, where $t \in \mathbb{R}$ is a real parameter.

(a) Find *all the values* of t such that the vector equation

$$x_1(6, 2) + x_2(4, t) = (1, 1)$$

has a solution.

(b) Find *all the values* of t such that the vector equation

$$x_1(6, 2) + x_2(4, t) = \mathbf{b}$$

always has a solution regardless of the choice of $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$.

(c) For which value(s) of t does the vector equation

$$x_1(6, 2) + x_2(4, t) = (9, 3)$$

have an *infinite number* of solutions?

Solution.

(a) As we have seen in class, the vector equation $x_1(6, 2) + x_2(4, t) = (1, 1)$ is equivalent to the system of linear equations

$$\begin{aligned} 6x_1 + 4x_2 &= 1 \\ 2x_1 + tx_2 &= 1. \end{aligned}$$

We start solving this system by multiplying the second equation by 3 and subtracting the result from the first equation. We obtain

$$4x_2 - 3tx_2 = -2. \tag{3}$$

If $4 - 3t \neq 0$, i.e., if $t \neq \frac{4}{3}$, we can solve for x_2 to obtain

$$x_2 = \frac{-2}{4 - 3t}.$$

Substituting this back into the first equation gives

$$6x_1 + 4\frac{-2}{4 - 3t} = 1$$

and so

$$x_1 = \frac{1}{6}\left(1 + \frac{8}{4-3t}\right) = \frac{1}{2}\left(\frac{4-t}{4-3t}\right).$$

Thus for all $t \in \mathbb{R}$, except for the single value $t = \frac{4}{3}$, the vector equation $x_1(6,2) + x_2(4,t) = (1,1)$ has the unique solution

$$x_1 = \frac{1}{2}\left(\frac{4-t}{4-3t}\right), \qquad x_2 = \frac{-2}{4-3t}.$$

On the other hand, when $t = \frac{4}{3}$, the vector equation has no solution since (3) gives $0 = -2$ in this case.

(b) Here, we give a geometric argument.

Recall that two vectors are linearly dependent if and only if one is a scalar multiple of the other. In $\mathbb{R}^2$ this happens if and only if the two vectors lie on the same line through the origin, that is, these two vectors are parallel. If $t = \frac{4}{3}$ then the vectors $(6,2)$ and $(4,t) = \left(4, \frac{4}{3}\right)$ are both on the line $y = \frac{1}{3}x$ and so the span of the vectors is that line and not all of $\mathbb{R}^2$. Thus in this case, it is not true that an arbitrary $\mathbf{b} \in \mathbb{R}^2$ is in the span of $(6,2)$ and $(4,t)$ and so the vector equation $x_1(6,2) + x_2(4,t) = \mathbf{b}$ does not always have a solution.

On the other hand if $t \neq \frac{4}{3}$ then the vectors $(6,2)$ and $(4,t)$ are not on the same line, that is, they are not parallel (one is not a scalar multiple of the other), and so they are linearly independent. These two linearly independent vectors therefore span all of $\mathbb{R}^2$ and so $x_1(6,2) + x_2(4,t) = \mathbf{b}$ has a (unique) solution for any $\mathbf{b} = (b_1, b_2) \in \mathbb{R}$.

(c) We have learned in class that a linear vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$ has infinitely many solutions if and only if $\mathbf{b}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ and the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

In our case ($\mathbf{v}_1 = (6,2)$, $\mathbf{v}_2 = (4,t)$, $\mathbf{b} = (9,3)$), we have $\mathbf{b} = \frac{3}{2}\mathbf{v}_1$, so $\mathbf{b} \in S_{(\mathbf{v}_1,\mathbf{v}_2)}$ regardless of the value of t. Also, from part (b), $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if $t = \frac{4}{3}$. Thus the vector equation $x_1(6,2) + x_2(4,t) = (9,3)$ has infinitely many solution when $t = \frac{4}{3}$ (and for all other values of t it has exactly one solution).