1. In (a), (b), and (c) below, the matrix on the right hand side is obtained from the one on the left by an elementary row operation. Identify the operation in each case.

(a)

$$\begin{bmatrix} 4 & 2 & 1 & 5 \\ -1 & 3 & 1 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 2 & 1 & 5 \\ 3 & -9 & -3 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -5 & 6 \\ 6 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 6 & 3 & 1 \\ 2 & -5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ -3 & 1 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 0 & -5 \\ 2 & -1 & 4 \\ -3 & 1 & 8 \end{bmatrix}$$

(d) Apply the three row operations you have identified, successively, in the same order, to the matrix below and write down the resulting matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 0 & 4 \\ -3 & 0 & 1 & 4 & 0 & 6 \\ 0 & 7 & 0 & 0 & 1 & -9 \end{bmatrix} \longrightarrow \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}$$

Solution.

(a) Multiply the second row by -3:

$$\begin{bmatrix} 4 & 2 & 1 & 5 \\ -1 & 3 & 1 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix} \xrightarrow{-3\cdot \text{R2}} \begin{bmatrix} 4 & 2 & 1 & 5 \\ 3 & -9 & -3 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

(b) Switch rows 1 and 3:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -5 & 6 \\ 6 & 3 & 1 \end{bmatrix} \xrightarrow{\text{R1} \leftrightarrow \text{R3}} \begin{bmatrix} 6 & 3 & 1 \\ 2 & -5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) Add -2 times row 2 to row 1 and put the result in row 1:

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ -3 & 1 & 8 \end{bmatrix} \xrightarrow{\text{R1}-2\cdot\text{R2} \to \text{R1}} \begin{bmatrix} -3 & 0 & -5 \\ 2 & -1 & 4 \\ -3 & 1 & 8 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 0 & 4 \\ -3 & 0 & 1 & 4 & 0 & 6 \\ 0 & 7 & 0 & 0 & 1 & -9 \end{bmatrix} \xrightarrow{-3\cdot\text{R2}} \begin{bmatrix} 1 & -2 & 0 & 3 & 0 & 4 \\ 9 & 0 & -3 & -12 & 0 & -18 \\ 0 & 7 & 0 & 0 & 1 & -9 \end{bmatrix}$$

$$\xrightarrow{\text{R1} \leftrightarrow \text{R3}} \begin{bmatrix} 0 & 7 & 0 & 0 & 1 & -9 \\ 9 & 0 & -3 & -12 & 0 & -18 \\ 1 & -2 & 0 & 3 & 0 & 4 \end{bmatrix} \xrightarrow{\text{R1}-2\cdot\text{R2} \to \text{R1}} \begin{bmatrix} -18 & 7 & 6 & 24 & 1 & 27 \\ 9 & 0 & -3 & -12 & 0 & -18 \\ 1 & -2 & 0 & 3 & 0 & 4 \end{bmatrix}$$

2. Use row operations to put the following matrices into RREF.

(a) $\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 10 & 6 & 9 \\ 15 & 9 & 12 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -6 & -1 & 6 \\ 4 & -12 & 2 & 20 \\ 3 & -9 & 0 & 12 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 6 & -1 & -1 \\ 3 & -18 & 1 & 0 \\ 2 & -12 & 3 & 0 \end{bmatrix}$

Solution. Here are the RREFs of the matrices, along with one possible path of calculation to arrive at the RREF. Other paths, as long as they contain no arithmetic mistakes, are equally valid, and will arrive at the same answer. Leading ones are in red to indicate the progress towards getting the matrix into RREF.

(a) $\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Possible Calculation:

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix} \xrightarrow{\text{R2-2 R1} \to \text{R2}} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{R1-2 R2} \to \text{R1}} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{R1/3}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(b) $\begin{bmatrix} 10 & 6 & 9 \\ 15 & 9 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Possible Calculation:

$$\begin{bmatrix} 10 & 6 & 9 \\ 15 & 9 & 12 \end{bmatrix} \xrightarrow{\text{R2/3}} \begin{bmatrix} 10 & 6 & 9 \\ 5 & 3 & 4 \end{bmatrix} \xrightarrow{\text{R1-2 R2} \to \text{R2}} \begin{bmatrix} 0 & 0 & 1 \\ 5 & 3 & 4 \end{bmatrix} \xrightarrow{\text{R2-4 R1} \to \text{R2}} \begin{bmatrix} 0 & 0 & 1 \\ 5 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{R1}\leftrightarrow\text{R2}} \begin{bmatrix} 5 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{R1/5}} \begin{bmatrix} 1 & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $\begin{bmatrix} 2 & -6 & -1 & 6 \\ 4 & -12 & 2 & 20 \\ 3 & -9 & 0 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Possible Calculation:

$$\begin{bmatrix} 2 & -6 & -1 & 6 \\ 4 & -12 & 2 & 20 \\ 3 & -9 & 0 & 12 \end{bmatrix} \xrightarrow{\text{R2-2 R1} \to \text{R2}} \begin{bmatrix} 2 & -6 & -1 & 6 \\ 0 & 0 & 4 & 8 \\ 3 & -9 & 0 & 12 \end{bmatrix} \xrightarrow{\text{R2/4}} \begin{bmatrix} 2 & -6 & -1 & 6 \\ 0 & 0 & 1 & 2 \\ 3 & -9 & 0 & 12 \end{bmatrix}$$

$$\xrightarrow{\text{R1+R2} \to \text{R1}} \begin{bmatrix} 2 & -6 & 0 & 8 \\ 0 & 0 & 1 & 2 \\ 3 & -9 & 0 & 12 \end{bmatrix} \xrightarrow{\text{R1/2}} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 3 & -9 & 0 & 12 \end{bmatrix} \xrightarrow{\text{R3-3 R1} \to \text{R3}} \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) $\begin{bmatrix} -1 & 6 & -1 & -1 \\ 3 & -18 & 1 & 0 \\ 2 & -12 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Possible Calculation:

$$\begin{bmatrix} -1 & 6 & -1 & -1 \\ 3 & -18 & 1 & 0 \\ 2 & -12 & 3 & 0 \end{bmatrix} \xrightarrow{\text{R2+3 R1} \to \text{R2}} \begin{bmatrix} -1 & 6 & -1 & -1 \\ 0 & 0 & -2 & -3 \\ 2 & -12 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{R3+2 R1} \to \text{R3}} \left[\begin{array}{cccc} -1 & 6 & -1 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{\text{R2+2 R3} \to \text{R2}} \left[\begin{array}{cccc} -1 & 6 & -1 & -1 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{\text{R2/(-7)}} \left[\begin{array}{cccc} -1 & 6 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{\text{R3+2 R2} \to \text{R3}} \left[\begin{array}{cccc} -1 & 6 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{R1+R2} \to \text{R1}} \left[\begin{array}{cccc} -1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{R1+R3} \to \text{R1}} \left[\begin{array}{cccc} -1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{R1}\cdot(-1)} \left[\begin{array}{cccc} 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{R2}\leftrightarrow\text{R3}} \left[\begin{array}{cccc} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

3. For each of the following matrices, already in RREF, parameterize all the solutions to the corresponding system of equations. Write your solution in vector form (in $\mathbb{R}^n$ where n is the number of unknown variables).

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 2 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & -4 & 0 & -\frac{3}{2} & 5 \\ 0 & 0 & 1 & 6 & \frac{1}{9} \end{array}\right]$

(c) $\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 1 & -5 \\ 0 & 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 & 7 & 4 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right]$

(e) $\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 3 & 0 & 11 & -7 \\ 0 & 0 & 1 & 4 & 0 & 6 & 21 \\ 0 & 0 & 0 & 0 & 1 & -9 & 14 \end{array}\right]$

Solution. The variable names in the following problems are arbitrary — a correct solution does not have to use the same names. The method in all cases is the same: The columns with leading ones correspond to the dependent variables; and the others are the independent variables. We assign t_1, t_2, etc to all the independent variables; each row of the matrix then lets us write one of the dependent variables in terms of the independent ones. Finally, we rewrite the parameterization in vector form (in $\mathbb{R}^n$ where n is the number of unknown variables).

(a) $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 8 \\ 0 & 1 & 2 & 2 \end{array}\right] \end{array}$

The dependent variables are x and y, and the independent variable is z. Setting $z = t$, we have $x = 8 - 3t$ and $y = 2 - 2t$, or, in vector form (in $\mathbb{R}^3$):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}.$$

(b) $\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & -4 & 0 & -\frac{3}{2} & 5 \\ 0 & 0 & 1 & 6 & \frac{1}{9} \end{array}\right] \end{array}$

The dependent variables are x and z, and the independent variables are y and w. Setting $y = t_1$ and $w = t_2$, the parameterization is $x = 5 + 4t_1 + \frac{3}{2}t_2$ and $z = \frac{1}{9} - 6t_2$. In vector form (in $\mathbb{R}^4$), this is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ \frac{1}{9} \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} \frac{3}{2} \\ 0 \\ -6 \\ 1 \end{pmatrix}.$$

(c)
$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|r} \color{red}{1} & 0 & 3 & 0 & 1 & -5 \\ 0 & \color{red}{1} & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & \color{red}{1} & 7 & 4 \end{array}\right] \end{array}$$

The dependent variables are x_1, x_2, and x_4, and the independent variables are x_3 and x_5. Setting $x_3 = t_1$, $x_5 = t_2$, we have $x_1 = -5 - 3t_1 - t_2$, $x_2 = 9 - 2t_1 - 8t_2$, and $x_4 = 4 - 7t_2$. Written in vector form (in $\mathbb{R}^5$) this is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ 0 \\ 4 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -8 \\ 0 \\ -7 \\ 1 \end{bmatrix}.$$

(d)
$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|r} \color{red}{1} & 0 & 0 & 8 \\ 0 & \color{red}{1} & 0 & 5 \\ 0 & 0 & \color{red}{1} & -3 \end{array}\right] \end{array}$$

All of the variables are dependent variables, and $x_1 = 8$, $x_2 = 5$, and $x_3 = -3$ is the unique solution. In vector form (in $\mathbb{R}^3$) this is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ -3 \end{pmatrix}.$$

(e)
$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{rrrrrr|r} \color{red}{1} & -2 & 0 & 3 & 0 & 11 & -7 \\ 0 & 0 & \color{red}{1} & 4 & 0 & 6 & 21 \\ 0 & 0 & 0 & 0 & \color{red}{1} & -9 & 14 \end{array}\right] \end{array}$$

The dependent variables are x_1, x_3, and x_5, and the independent variables are x_2, x_4, and x_6. Setting $x_2 = t_1$, $x_4 = t_2$, and $x_6 = t_3$, we have $x_1 = -7 + 2t_1 - 3t_2 - 11t_3$, $x_3 = 21 - 4t_2 - 6t_3$, and $x_5 = 14 + 9t_3$. In vector form (in $\mathbb{R}^6$) this is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 21 \\ 0 \\ 14 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -11 \\ 0 \\ -6 \\ 0 \\ 9 \\ 1 \end{bmatrix}.$$

4. Parameterize all solutions to each of the systems of linear equations below, writing the parameterization in vector form. (That is, for each of the systems below, go through the steps of encoding the system in a matrix, using row operations to put each matrix in RREF, and writing down the general solution from the RREF form of the matrix.)

(1)
$$\left\{ \begin{array}{rcrcrcrcr} 4x & + & 3y & + & 6z & + & 5w & = & -7 \\ & - & y & + & 2z & + & 2w & = & -9 \\ x & + & 4y & - & 5z & - & 5w & = & 27 \end{array} \right.$$

$$(2)\quad \begin{cases} 2x + 2y + 3z + 4w = 2 \\ -x + y + 2z + 7w = 1 \\ -3x + z + 5w = 1 \\ 6x + 2y + 1z + 3w = -1 \end{cases}$$

$$(3)\quad \begin{cases} 3x_1 + 9x_3 + 6x_4 = 3 \\ 2x_1 + x_2 + 5x_3 + 2x_4 = 1 \\ 5x_1 + 8x_2 + 7x_3 - 6x_4 = -3 \end{cases}$$

Solution. For each of the systems of equations, we go through the three steps above.

(a) encode the system of linear equations in a augmented matrix;

(b) use elementary row operations to put each matrix in RREF;

(c) write down the general solution from the RREF form of the matrix.

$$(1)\quad \begin{cases} 4x + 3y + 6z + 5w = -7 \\ -y + 2z + 2w = -9 \\ x + 4y - 5z - 5w = 27 \end{cases}$$

(a)
$$\begin{array}{c} \begin{matrix} x & y & z & w & \end{matrix} \\ \left[\begin{array}{cccc|c} 4 & 3 & 6 & 5 & -7 \\ 0 & -1 & 2 & 2 & -9 \\ 1 & 4 & -5 & -5 & 27 \end{array}\right] \end{array}$$

(b)
$$\left[\begin{array}{cccc|c} 4 & 3 & 6 & 5 & -7 \\ 0 & -1 & 2 & 2 & -9 \\ 1 & 4 & -5 & -5 & 27 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -3 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Possible Calculation:

$$\left[\begin{array}{cccc|c} 4 & 3 & 6 & 5 & -7 \\ 0 & -1 & 2 & 2 & -9 \\ 1 & 4 & -5 & -5 & 27 \end{array}\right] \xrightarrow{R3+4R2\to R3} \left[\begin{array}{cccc|c} 4 & 3 & 6 & 5 & -7 \\ 0 & -1 & 2 & 2 & -9 \\ 1 & 0 & 3 & 3 & -9 \end{array}\right]$$

$$\xrightarrow{R1+3R2\to R1} \left[\begin{array}{cccc|c} 4 & 0 & 12 & 11 & -34 \\ 0 & -1 & 2 & 2 & -9 \\ 1 & 0 & 3 & 3 & -9 \end{array}\right] \xrightarrow{R1-4R2\to R1} \left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 2 \\ 0 & -1 & 2 & 2 & -9 \\ 1 & 0 & 3 & 3 & -9 \end{array}\right]$$

$$\xrightarrow{R2+2R1\to R2} \left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 2 \\ 0 & -1 & 2 & 0 & -5 \\ 1 & 0 & 3 & 3 & -9 \end{array}\right] \xrightarrow{R1\cdot(-1);R2\cdot(-1)} \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 5 \\ 1 & 0 & 3 & 3 & -9 \end{array}\right]$$

$$\xrightarrow{\text{Reorder rows}} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 3 & -9 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R1-3R3\to R1} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -3 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

(c)
$$\begin{array}{c} \begin{matrix} x & y & z & w & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -3 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \end{array}$$

The dependent variables are x, y, and w and the independent variable is z. Setting $z = t$, we have $x = -3 - 3t$, $y = 5 + 2t$, and $z = -2$, which in vector form (in $\mathbb{R}^4$) is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

(2) $\begin{cases} 2x + 2y + 3z + 4w = 2 \\ -x + y + 2z + 7w = 1 \\ -3x + z + 5w = 1 \\ 6x + 2y + 1z + 3w = -1 \end{cases}$

(a) $$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 2 & 2 & 3 & 4 & 2 \\ -1 & 1 & 2 & 7 & 1 \\ -3 & 0 & 1 & 5 & 1 \\ 6 & 2 & 1 & 3 & -1 \end{array}\right] \end{array}$$

(b) $$\left[\begin{array}{cccc|c} 2 & 2 & 3 & 4 & 2 \\ -1 & 1 & 2 & 7 & 1 \\ -3 & 0 & 1 & 5 & 1 \\ 6 & 2 & 1 & 3 & -1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Possible Calculation:

$$\left[\begin{array}{cccc|c} 2 & 2 & 3 & 4 & 2 \\ -1 & 1 & 2 & 7 & 1 \\ -3 & 0 & 1 & 5 & 1 \\ 6 & 2 & 1 & 3 & -1 \end{array}\right] \xrightarrow{R1+2R2\to R1} \left[\begin{array}{cccc|c} 0 & 4 & 7 & 18 & 4 \\ -1 & 1 & 2 & 7 & 1 \\ -3 & 0 & 1 & 5 & 1 \\ 6 & 2 & 1 & 3 & -1 \end{array}\right] \xrightarrow{R4+2R3\to R4} \left[\begin{array}{cccc|c} 0 & 4 & 7 & 18 & 4 \\ -1 & 1 & 2 & 7 & 1 \\ -3 & 0 & 1 & 5 & 1 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right]$$

$$\xrightarrow{R1-2R4\to R1} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ -1 & 1 & 2 & 7 & 1 \\ -3 & 0 & 1 & 5 & 1 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right] \xrightarrow{R2\cdot(-1)} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & -1 & -2 & -7 & -1 \\ -3 & 0 & 1 & 5 & 1 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right]$$

$$\xrightarrow{R3+3R2\to R3} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & -1 & -2 & -7 & -1 \\ 0 & -3 & -5 & -16 & -2 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right] \xrightarrow{R3+R4\to R3} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & -1 & -2 & -7 & -1 \\ 0 & -1 & -2 & -3 & -1 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right]$$

$$\xrightarrow{R3\cdot(-1)} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & -1 & -2 & -7 & -1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right] \xrightarrow{R2+R3\to R2} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 2 & 3 & 13 & 1 \end{array}\right]$$

$$\xrightarrow{R4-2R3\to R4} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & 7 & -1 \end{array}\right] \xrightarrow{R4+R1\to R4} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R4\cdot(-1)} \left[\begin{array}{cccc|c} 0 & 0 & 1 & -8 & 2 \\ 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R1+8R4\to R1} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -6 \\ 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\text{Reorder}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R1+4R4\to R1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R2-3R4\to R2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R2-2R3\to R2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

(c) $\begin{array}{c} \begin{matrix} x & y & z & w & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \end{array}$

All of the variables are dependent variables, and $x = -4$, $y = 16$, $z = -6$, and $w = -1$ is the unique solution. In vector form (in $\mathbb{R}^4$) this is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -4 \\ 16 \\ -6 \\ -1 \end{pmatrix}.$$

(3) $\left\{ \begin{array}{rcrcrcrcr} 3x_1 & & & + & 9x_3 & + & 6x_4 & = & 3 \\ 2x_1 & + & x_2 & + & 5x_3 & + & 2x_4 & = & 1 \\ 5x_1 & + & 8x_2 & + & 7x_3 & - & 6x_4 & = & -3 \end{array} \right.$

(a) $\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 3 & 0 & 9 & 6 & 3 \\ 2 & 1 & 5 & 2 & 1 \\ 5 & 8 & 7 & -6 & -3 \end{array}\right] \end{array}$

(b) $\left[\begin{array}{cccc|c} 3 & 0 & 9 & 6 & 3 \\ 2 & 1 & 5 & 2 & 1 \\ 5 & 8 & 7 & -6 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Possible Calculation:

$$\left[\begin{array}{cccc|c} 3 & 0 & 9 & 6 & 3 \\ 2 & 1 & 5 & 2 & 1 \\ 5 & 8 & 7 & -6 & -3 \end{array}\right] \xrightarrow{R1/3} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 5 & 2 & 1 \\ 5 & 8 & 7 & -6 & -3 \end{array}\right] \xrightarrow{R2-2R1\to R2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 5 & 8 & 7 & -6 & -3 \end{array}\right]$$

$$\xrightarrow{R4-5R1\to R4} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 0 & 8 & -8 & -16 & -8 \end{array}\right] \xrightarrow{R4-8R2\to R4} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) $\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

The dependent variables are x_1 and x_2, and the independent variables are x_3 and x_4. Setting $x_3 = t_1$ and $x_4 = t_2$, we have $x_1 = 1 - 3t_1 - 2t_2$ and $x_2 = -1 + t_1 + 2t_2$, or in vector form (in $\mathbb{R}^4$), we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$