1. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1,0,0), \quad \mathbf{v}_2 = (1,1,0), \quad \mathbf{v}_3 = (1,2,1), \quad \mathbf{v}_4 = (0,0,3).$$

   (a) Show that $\{\mathbf{v}_2, \mathbf{v}_3\}$ is not a generating set for $\mathbb{R}^3$.

   (b) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a generating set for $\mathbb{R}^3$.

   (c) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a generating set for $\mathbb{R}^3$.

2. Consider the subset $\mathbf{W}$ of $\mathbb{R}^4$ given by

$$\mathbf{W} = \{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w = 0,\ y = z\}.$$

It is easy to show that $\mathbf{W}$, with the usual addition and scalar multiplication operations, is a subspace of $\mathbb{R}^4$. Let

$$\mathbf{v}_1 = (1,0,0,-1), \qquad \mathbf{v}_2 = (1,1,1,-3).$$

Show that $B = (\mathbf{v}_1, \mathbf{v}_2)$ is a basis for $\mathbf{W}$ and compute the coordinates of $\mathbf{u} = (-6,6,6,-6)$ with respect to B.

3. Suppose $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is a basis for a vector space $\mathbf{V}$ and let

$$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{u}_2 = \mathbf{v}_2 + \mathbf{v}_3, \quad \mathbf{u}_3 = \mathbf{v}_3 + \mathbf{v}_4, \quad \mathbf{u}_4 = \mathbf{v}_4. \tag{$*$}$$

Prove that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is also a basis for $\mathbf{V}$.

4. Let $\mathbf{V}$ be a finite-dimensional vector space and let $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ be a basis for $\mathbf{V}$. Suppose $\mathbf{v}$ is an arbitrary *nonzero* vector in $\mathbf{V}$. Show that there is a vector $\mathbf{v}_i$ in B such that if we replace $\mathbf{v}_i$ with $\mathbf{v}$ in B, the resulting m-tuple of vectors is still a basis for $\mathbf{V}$.

5. Let $\mathbf{V}$ be a finite-dimensional vector space and let $\mathbf{W}$ be a subspace of $\mathbf{V}$.

   (a) Show that $\dim\mathbf{W} \le \dim\mathbf{V}$.

   (b) Suppose $\dim\mathbf{W} = \dim\mathbf{V}$. Show that $\mathbf{W} = \mathbf{V}$.

Hint: The Key Lemma may be useful in part (a).

6. Consider the subspace $\mathbf{W}$ of $\mathbb{R}^4$ defined by

$$\mathbf{W} = \{(x,y,z,w) \in \mathbb{R}^4 : x+2w = 0,\ 2z+w = 0\}.$$

Show that $\mathbf{W}$ has dimension 2.

7. Consider the subspace $\mathbf{W}$ of $\mathbb{R}^4$ defined by

$$\mathbf{W} = \{(x,y,z,w) \in \mathbb{R}^4 : x-y = 0,\ z+w = 0,\ y+w = 0\}.$$

Show that $\mathbf{W}$ has dimension 1.

8. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomial functions (from $\mathbb{R}$ to $\mathbb{R}$) of degree $\leqslant 2$, with real coefficients. That is, the polynomials of the form

$$a_0 + a_1 x + a_2 x^2, \qquad x \in \mathbb{R},$$

with a_0, a_1, a_2, $\in \mathbb{R}$. The operations on $\mathcal{P}_2(\mathbb{R})$ are the usual addition and scalar multiplication of polynomials.

(a) Show that $\mathcal{B} = (f_1, f_2, f_3)$ is a basis for $\mathcal{P}_2(\mathbb{R})$, where

$$f_1(x) = 1, \quad f_2(x) = x - 2, \quad f_3(x) = (x-2)^2, \quad x \in \mathbb{R},$$

(b) What is the dimension of $\mathcal{P}_2(\mathbb{R})$?

(c) Given $f \in \mathcal{P}_2(\mathbb{R})$, where

$$f(x) = 3x^2 + x + 9, \quad x \in \mathbb{R},$$

find the coordinates of f in terms of basis $\mathcal{B}$.