1. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1,0,0), \quad \mathbf{v}_2 = (1,1,0), \quad \mathbf{v}_3 = (1,2,1), \quad \mathbf{v}_4 = (0,0,3).$$

(a) Show that $\{\mathbf{v}_2, \mathbf{v}_3\}$ is not a generating set for $\mathbb{R}^3$.

(b) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a generating set for $\mathbb{R}^3$.

(c) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a generating set for $\mathbb{R}^3$.

Solution.

(a) The span of $\mathbf{v}_2$ and $\mathbf{v}_3$ is the collection of all linear combinations

$$\alpha\mathbf{v}_2 + \beta\mathbf{v}_3 = \alpha(1,1,0) + \beta(1,2,1) = (\alpha+\beta, \alpha+2\beta, \beta)$$

for $\alpha, \beta \in \mathbb{R}$. Since $(\alpha+\beta)+\beta = \alpha+2\beta$, any (x,y,z) in the span of $\{\mathbf{v}_2, \mathbf{v}_3\}$ must satisfy $x+z=y$, which means that, for example, the vector $(1,1,1)$ is not in the span. Thus $\{\mathbf{v}_2, \mathbf{v}_3\}$ does not generate $\mathbb{R}^3$. We can also conclude from the dimension of $\mathbb{R}^3$ that the set with only two vectors cannot generate the entire vector space $\mathbb{R}^3$.

(b) We have to show that any $(x,y,z) \in \mathbb{R}^3$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, i.e., for any $x,y,z \in \mathbb{R}$ the vector equation

$$\alpha(1,0,0) + \beta(1,1,0) + \gamma(1,2,1) = (x,y,z)$$

has a solution for α, β, γ. This vector equation is equivalent to the system of linear equations

$$\begin{array}{rcl} \alpha+\beta+\gamma & = & x \\ \beta+2\gamma & = & y \\ \gamma & = & z \end{array}.$$

From the last equation we get $\gamma = z$. Plugging this value into the second equation gives $\beta + 2z = y$, i.e., $\beta = y - 2z$. Finally, from this and the first equation we obtain $\alpha + (y-2z) + z = x$, which gives $\alpha = x - y + z$. Thus the system has a (unique) solution $\alpha = x-y+z$, $\beta = y-2z$, and $\gamma = z$, which means that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ generates $\mathbb{R}^3$.

(c) Clearly, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ generates $\mathbb{R}^3$ since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ generates $\mathbb{R}^3$. More formally, since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, we have $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} \subset S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)} \subset \mathbb{R}^3$ and we know that $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} = \mathbb{R}^3$, so $S_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)} = \mathbb{R}^3$.

2. Consider the subset $\mathbf{W}$ of $\mathbb{R}^4$ given by

$$\mathbf{W} = \{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w = 0,\ y = z\}.$$

It is easy to show that $\mathbf{W}$, with the usual addition and scalar multiplication operations, is a subspace of $\mathbb{R}^4$. Let

$$\mathbf{v}_1 = (1,0,0,-1), \qquad \mathbf{v}_2 = (1,1,1,-3).$$

Show that $B = (\mathbf{v}_1, \mathbf{v}_2)$ is a basis for $\mathbf{W}$ and compute the coordinates of $\mathbf{u} = (-6,6,6,-6)$ with respect to B.

Solution. Clearly, the entries of $\mathbf{v}_1 = (1,0,0,-1)$ and $\mathbf{v}_2 = (1,1,1,-3)$ satisfy the conditions $x+y+z+w=0$ and $y=z$, so $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{W}$. To show that $(\mathbf{v}_1, \mathbf{v}_2)$ is basis for $\mathbf{W}$, we have to show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a *generating set* for $\mathbf{W}$, and that $\mathbf{v}_1$ and $\mathbf{v}_2$ are *linearly independent*. We start with linear independence, which is easier to show.

Linear independence. We know from class that two vectors in a vector space are linearly dependent if and only if one of them is a scalar multiple of the other. Here the two vectors are $\mathbf{v}_1 = (1, 0, 0, -1)$ and $\mathbf{v}_2 = (1, 1, 1, -3)$. Clearly, neither is a scalar multiple of the other one, so they are *linearly independent*.

Generating set. We have to show that $\mathbf{v} \in \mathbf{W}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $\mathbf{v} = (x, y, z, w) \in \mathbf{W}$. Then, by the definition of $\mathbf{W}$, we know that $y = z$, so we can rewrite $\mathbf{v}$ as $\mathbf{v} = (x, y, y, w)$. Also by the definition of $\mathbf{W}$ we know that the sum of the entries of $\mathbf{v}$ must be zero: $x+y+y+w = 0$, which is equivalent to $w = -x - 2y$. Thus every vector $\mathbf{v}$ in $\mathbf{W}$ is of the form

$$\mathbf{v} = (x, y, y, -x - 2y)$$

for some $x, y \in \mathbb{R}$. Conversely, every $\mathbf{v}$ of this form is in $\mathbf{W}$. We want to show that any such vector can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Letting $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 &= \alpha(1, 0, 0, -1) + \beta(1, 1, 1, -3) \\ &= (\alpha + \beta, \beta, \beta, -\alpha - 3\beta), \end{aligned}$$

so we have to see that the vector equation

$$(\alpha + \beta, \beta, \beta, -\alpha - 3\beta) = (x, y, y, -x - 2y)$$

can be solved for α and β when $x, y \in \mathbb{R}$ are arbitrary. The vector equation is equivalent to the following system of linear equations with unknowns α and β,

$$\begin{array}{ccc} \alpha + \beta & = & x \\ \beta & = & y \\ \beta & = & y \\ -\alpha - 3\beta & = & -x - 2y \end{array}$$

(the second and third equations are identical). From the second (or third) equation we get $\beta = y$. Substituting this into the first equation gives $\alpha + y = x$, i.e., $\alpha = x - y$. Finally, we check that the third equation is satisfied when setting $\alpha = x - y$ and $\beta = y$. Thus the system has a (unique) solution and every vector $\mathbf{v} = (x, y, y, -x - 2y) \in \mathbf{W}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ as

$$\begin{aligned} (x, y, y, -x - 2y) &= (x - y)\mathbf{v}_1 + y\mathbf{v}_2 \\ &= (x - y)(1, 0, 0, -1) + y(1, 1, 1, -3) \end{aligned}$$

which proves that $\{\mathbf{v}_2, \mathbf{v}_2\}$ is a *generating set* for $\mathbf{W}$.

We have shown that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and they generate $\mathbf{W}$, so we conclude that $B = (\mathbf{v}_1, \mathbf{v}_2)$ is a basis for $\mathbf{W}$.

Finally, for the vector $(-6, 6, 6, -6) \in \mathbf{W}$ we have $x = -6$ and $y = 6$, so the formula above yields the coordinate vector of $(-6, 6, 6, -6)$ in the basis B as

$$(-6 - 6, 6)_B = (-12, 6)_B.$$

3. Suppose $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is a basis for a vector space $\mathbf{V}$ and let

$$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{u}_2 = \mathbf{v}_2 + \mathbf{v}_3, \quad \mathbf{u}_3 = \mathbf{v}_3 + \mathbf{v}_4, \quad \mathbf{u}_4 = \mathbf{v}_4. \qquad (*)$$

Prove that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is also a basis for $\mathbf{V}$.

Solution. Let

$$B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4), \qquad B' = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4).$$

In order to show that B' is a basis for $\mathbf{V}$, we have to show that the vectors in B' generate $\mathbf{V}$ and that these vectors are also linearly independent (this a the definition of a basis for $\mathbf{V}$). We do this in one step: we show that every $\mathbf{v} \in \mathbf{V}$ can be written as a linear combination of the vectors in B' in a *unique way*, which proves that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a linearly independent generating set for $\mathbf{V}$.

First we note that the definition of B' in $(*)$ implies that an arbitrary linear combination $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\alpha_4\mathbf{u}_4$ of the vectors in B' is also a linear combination of the vectors in B:

$$\begin{aligned}\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\alpha_4\mathbf{u}_4 &= \alpha_1(\mathbf{v}_1+\mathbf{v}_2)+\alpha_2(\mathbf{v}_2+\mathbf{v}_3)+\alpha_3(\mathbf{v}_3+\mathbf{v}_4)+\alpha_4\mathbf{v}_4 \\ &= \alpha_1\mathbf{v}_1+(\alpha_1+\alpha_2)\mathbf{v}_2+(\alpha_2+\alpha_3)\mathbf{v}_3+(\alpha_3+\alpha_4)\mathbf{v}_4. \end{aligned} \tag{1}$$

Since B is a basis, for any $\mathbf{v}\in\mathbf{V}$ there exist unique real numbers $\beta_1,\beta_2,\beta_3,\beta_4$ (the coordinates of $\mathbf{v}$ with respect to the basis B) such that

$$\mathbf{v}=\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2+\beta_3\mathbf{v}_3+\beta_4\mathbf{v}_4. \tag{2}$$

Comparing the coefficients of the vectors $\mathbf{v}_i$ in equations (1) and (2), we obtain that we can write $\mathbf{v}$ as a linear combination of the vectors in B' if and only if there exist $\alpha_1,\alpha_2,\alpha_2,\alpha_4$ that solve the system of linear equations

$$\begin{aligned}\alpha_1 &= \beta_1\\ \alpha_1+\alpha_2 &= \beta_2\\ \alpha_2+\alpha_3 &= \beta_3\\ \alpha_3+\alpha_4 &= \beta_4.\end{aligned}$$

From the first equation we have $\alpha_1=\beta_1$. Plugging this into the second equation, we obtain $\alpha_2=\beta_2-\beta_1$. Substituting this into the third equation gives $\alpha_3=\beta_3-\beta_2+\beta_1$, and finally plugging this value into the last equations yields $\alpha_4=\beta_4-\beta_3+\beta_2-\beta_1$. Thus the system has the *unique* solution

$$\alpha_1=\beta_1,\quad \alpha_2=\beta_2-\beta_1,\quad \alpha_3=\beta_3-\beta_2+\beta_1,\quad \alpha_4=\beta_4-\beta_3+\beta_2-\beta_1.$$

In summary, we have proved that any vector $\mathbf{v}$ in $\mathbf{V}$ can be written as the linear combination of vectors in B', so the vectors in B' are a generating set for $\mathbf{V}$, and that this linear combination is unique for any $\mathbf{v}$, and so the vectors in B' are linearly independent. This proves that $B'=(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4)$ is a basis for $\mathbf{V}$.

4. Let $\mathbf{V}$ be a finite-dimensional vector space and let $B=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m)$ be a basis for $\mathbf{V}$. Suppose $\mathbf{v}$ is an arbitrary *nonzero* vector in $\mathbf{V}$. Show that there is a vector $\mathbf{v}_i$ in B such that if we replace $\mathbf{v}_i$ with $\mathbf{v}$ in B, the resulting m-tuple of vectors is still a basis for $\mathbf{V}$.

Solution. Let $\mathbf{v}\in\mathbf{V}$ be arbitrary such that $\mathbf{v}\neq\mathbf{0}$. Since B is a basis, $\mathbf{v}$ can be written as linear combination

$$\mathbf{v}=\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_m\mathbf{v}_m$$

for some $\alpha_1,\ldots,\alpha_m\in\mathbb{R}$. Since $\mathbf{v}\neq\mathbf{0}$, at least one α_i must be nonzero. To simplify the notation we assume that $\alpha_1\neq 0$. We claim that if we replace $\mathbf{v}_1$ with $\mathbf{v}$ in B, then the resulting list of m vectors is also a basis for $\mathbf{V}$, i.e., that $B'=(\mathbf{v},\mathbf{v}_2,\ldots,\mathbf{v}_m)$ is a basis.

In order to show that B' is a basis, we will prove that B' generates $\mathbf{V}$ and that the vectors in B' are linearly independent.

<u>Generating set.</u> Since $\alpha_1\neq 0$, $\mathbf{v}_1$ can be written as

$$\mathbf{v}_1=\frac{1}{\alpha_1}\mathbf{v}-\frac{\alpha_2}{\alpha_1}\mathbf{v}_2-\cdots-\frac{\alpha_m}{\alpha_1}\mathbf{v}_m. \tag{3}$$

Since B is a basis, any $\mathbf{u}\in\mathbf{V}$ can be written as a linear combination

$$\mathbf{u}=\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2+\cdots+\beta_m\mathbf{v}_m.$$

In view of (3), this gives

$$\begin{aligned}\mathbf{u} &= \beta_1\Big(\frac{1}{\alpha_1}\mathbf{v}-\frac{\alpha_2}{\alpha_1}\mathbf{v}_2-\cdots-\frac{\alpha_m}{\alpha_1}\mathbf{v}_m\Big)+\beta_2\mathbf{v}_2+\cdots+\beta_m\mathbf{v}_m\\ &= \frac{\beta_1}{\alpha_1}\mathbf{v}+\Big(\beta_2-\frac{\alpha_2}{\alpha_1}\Big)\mathbf{v}_2+\cdots+\Big(\beta_m-\frac{\alpha_m}{\alpha_1}\Big)\mathbf{v}_m.\end{aligned}$$

This proves that any $\mathbf{u}\in\mathbf{V}$ can be written as a linear combination of vectors in B', so B' generates $\mathbf{V}$.

Linear independence. Let $\gamma_1, \ldots, \gamma_m$ be coefficients such that

$$\mathbf{0} = \gamma_1 \mathbf{v} + \gamma_2 \mathbf{v}_2 + \cdots + \cdots \gamma_m \mathbf{v}_m.$$

To prove the linear independence of $\{\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, we have to prove that $\gamma_i = 0$ for all $i = 1, \ldots, m$. Using the fact that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_m \mathbf{v}_m$, we can rewrite the above as

$$\begin{aligned} \mathbf{0} &= \gamma_1 \big(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_m \mathbf{v}_m\big) + \gamma_2 \mathbf{v}_2 + \cdots + \gamma_m \mathbf{v}_m \\ &= \gamma_1 \alpha_1 \mathbf{v}_1 + \big(\gamma_2 + \gamma_1 \alpha_2\big) \mathbf{v}_2 + \cdots + \big(\gamma_m + \gamma_1 \alpha_m\big) \mathbf{v}_m. \end{aligned}$$

Since B is a basis, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a linearly independent set, so we must have $\gamma_1 \alpha_1 = 0$, $\gamma_2 + \gamma_1 \alpha_2 = 0$, $\ldots$, $\gamma_m + \gamma_1 \alpha_m = 0$. Since $\alpha_1 \neq 0$, $\gamma_1 \alpha_1 = 0$ implies that $\gamma_1 = 0$. But then $\gamma_2 + \gamma_1 \alpha_2 = 0$ implies $\gamma_2 = 0$. In general, $\gamma_1 = 0$ and $\gamma_i + \gamma_1 \alpha_i = 0$ give $\gamma_i = 0$ for all $i = 2, \ldots, m$. Thus $\gamma_1 = \gamma_2 = \ldots = \gamma_m = 0$, so $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a linearly independent set.

We have proved that the vectors in B' both generate $\mathbf{V}$ and are linearly independent, which means that B' is a basis for $\mathbf{V}$.

5. Let $\mathbf{V}$ be a finite-dimensional vector space and let $\mathbf{W}$ be a subspace of $\mathbf{V}$.

(a) Show that $\dim \mathbf{W} \leq \dim \mathbf{V}$.

(b) Suppose $\dim \mathbf{W} = \dim \mathbf{V}$. Show that $\mathbf{W} = \mathbf{V}$.

Hint: The Key Lemma may be useful in part (a).

Solution.

1. Recall the Key Lemma: In a finite dimensional vector space, the size of any linearly independent set is at most the size of any generating set.

   Let $n = \dim \mathbf{V}$ and let $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be a basis for $\mathbf{V}$. Similarly, let $m = \dim \mathbf{W}$ and let $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ be a basis for $\mathbf{W}$. Since $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a basis, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a generating set for $\mathbf{V}$. On the other hand, since $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ is a basis for $\mathbf{W}$ and $\mathbf{W} \subset \mathbf{V}$, $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is a linearly independent set in $\mathbf{V}$.

   Then the Key Lemma gives $m \leq n$, so $\dim \mathbf{W} \leq \dim \mathbf{V}$.

2. Now assume that $\dim \mathbf{W} = \dim \mathbf{V}$. We will show that $\mathbf{V} \subset \mathbf{W}$ in this case (by proving that any basis of $\mathbf{W}$ is also a basis for $\mathbf{V}$); this will imply that $\mathbf{W} = \mathbf{V}$ (since $\mathbf{W}$ is a subspace of $\mathbf{V}$).

   Let $n = \dim \mathbf{W} = \dim \mathbf{V}$, let $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ be a basis for $\mathbf{W}$, and let $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ be a basis for $\mathbf{V}$. If $\mathbf{v}$ is an arbitrary vector in $\mathbf{V}$, we claim that the set of $n+1$ vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \mathbf{v}\}$ is linearly dependent. This is true since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a generating set for $\mathbf{V}$ and by the Key Lemma the number of vectors in any linearly independent set cannot be greater than n.

   Now since $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \mathbf{v}\}$ is linearly dependent, if $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$ are coefficients such that

   $$\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \cdots + \alpha_n \mathbf{w}_n + \alpha_{n+1} \mathbf{v} = \mathbf{0},$$

   then at least one α_i is not zero. If we had $\alpha_{n+1} = 0$, then we would get $\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \cdots + \alpha_n \mathbf{w}_n = \mathbf{0}$, where at least one of $\alpha_1, \alpha_2, \ldots, \alpha_n$ is not zero. But this is impossible since $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$ is a basis and so $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a linearly independent set. Thus we must have $\alpha_{n+1} \neq 0$, so we obtain

   $$\mathbf{v} = \frac{-\alpha_1}{\alpha_{n+1}} \mathbf{w}_1 - \frac{\alpha_2}{\alpha_{n+1}} \mathbf{w}_2 - \cdots - \frac{\alpha_n}{\alpha_{n+1}} \mathbf{w}_n,$$

   which shows that $\mathbf{v}$ is in the span of $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$. Since $\mathbf{v} \in \mathbf{V}$ was arbitrary, we obtain $\mathbf{V} \subset \mathbf{W} = S_{(\mathbf{w}_1, \ldots, \mathbf{w}_n)}$. On the other hand, since $\mathbf{W}$ is a subspace of $\mathbf{V}$, we have $\mathbf{W} \subset \mathbf{V}$, and we conclude that $\mathbf{W} = \mathbf{V}$.

6. Consider the subspace $\mathbf{W}$ of $\mathbb{R}^4$ defined by

$$\mathbf{W} = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2w = 0,\ 2z + w = 0\}.$$

Show that $\mathbf{W}$ has dimension 2.

Solution. We will find a basis for $\mathbf{W}$ consisting of two vectors, which will give the desired result $\dim\mathbf{W} = 2$. Let $(x, y, z, w) \in \mathbf{W}$ be arbitrary. The requirements $x + 2w = 0$ and $2z + w = 0$ are the same as $x = -2w$ and $w = -2z$. Equivalently, $x = 4z$ and $w = -2z$, so any $\mathbf{v} = (x, y, z, w) \in \mathbf{W}$ has the form $\mathbf{v} = (4z, y, z, -2z)$. Conversely, any vector $(4z, y, z, -2z)$ with $y, z \in \mathbb{R}$ is in $\mathbf{W}$ (check this!), so we obtain

$$\mathbf{W} = \Big\{(4z, y, z, -2z) \in \mathbb{R}^4 \ : \ y, z \in \mathbb{R}\Big\}.$$

Let $\mathbf{w}_1 = (4, 0, 1, -2)$ and $\mathbf{w}_2 = (0, 1, 0, 0)$; then for any $y, z \in \mathbb{R}$,

$$(4z, y, z, -2z) = z(4, 0, 1, -2) + y(0, 1, 0, 0) = z\mathbf{w}_1 + y\mathbf{w}_2,$$

which means that $\{\mathbf{w}_1, \mathbf{w}_2\}$ generates $\mathbf{W}$. Since neither $\mathbf{w}_1$ nor $\mathbf{w}_2$ is a scalar multiple of the other vector, they are linearly independent. Thus $(\mathbf{w}_1, \mathbf{w}_2)$ is a basis for $\mathbf{W}$ and therefore $\mathbf{W}$ has dimension 2.

7. Consider the subspace $\mathbf{W}$ of $\mathbb{R}^4$ defined by

$$\mathbf{W} = \{(x, y, z, w) \in \mathbb{R}^4 : x - y = 0,\ z + w = 0,\ y + w = 0\}.$$

Show that $\mathbf{W}$ has dimension 1.

Solution. Proceeding as in the previous problem, we will find a basis for $\mathbf{W}$ consisting of a single vector, which shows $\dim\mathbf{W} = 1$. Let $(x, y, z, w) \in \mathbf{W}$ be arbitrary. The requirements $x - y = 0$ and $y + w = 0$ give $x = y$ and $w = -y$. But then the third condition $z + w = 0$ gives $z = y$. Thus any $(x, y, z, w) \in \mathbf{W}$ is of the form $(y, y, y, -y)$. Conversely, any vector $(y, y, y, -y)$ with $y \in \mathbb{R}$ is in $\mathbf{W}$, so

$$\mathbf{W} = \Big\{(y, y, y, -y) \in \mathbb{R}^4 : y \in \mathbb{R}\Big\}.$$

Since $(y, y, y, -y) = y(1, 1, 1, -1)$, we obtain that $\mathbf{W}$ is spanned by the single vector $\mathbf{w}_1 = (1, 1, 1, -1)$, and thus $\{\mathbf{w}_1\}$ is a generating set for $\mathbf{W}$. Since $\mathbf{w}_1 \neq \mathbf{0}$, $\{\mathbf{w}_1\}$ is also a linearly independent set, so $(\mathbf{w}_1)$ is a basis for $\mathbf{W}$, which means that $\mathbf{W}$ has dimension 1.

8. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomial functions (from $\mathbb{R}$ to $\mathbb{R}$) of degree $\leqslant 2$, with real coefficients. That is, the polynomials of the form

$$a_0 + a_1x + a_2x^2, \qquad x \in \mathbb{R},$$

with a_0, a_1, a_2, $\in \mathbb{R}$. The operations on $\mathcal{P}_2(\mathbb{R})$ are the usual addition and scalar multiplication of polynomials.

(a) Show that $\mathcal{B} = \{f_1, f_2, f_3\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$, where

$$f_1(x) = 1, \quad f_2(x) = x - 2, \quad f_3(x) = (x - 2)^2, \quad x \in \mathbb{R},$$

(b) What is the dimension of $\mathcal{P}_2(\mathbb{R})$?

(c) Given $f \in \mathcal{P}_2(\mathbb{R})$, where

$$f(x) = 3x^2 + x + 9, \quad x \in \mathbb{R},$$

find the coordinates of f in terms of basis $\mathcal{B}$.

Solution.

a) To show that $\mathcal{B} = (f_1, f_2, f_3)$ is a basis for $\mathcal{P}_2(\mathbb{R})$, we have to show that $\{f_1, f_2, f_3\}$ is both a *linearly independent set* in $\mathcal{P}_2(\mathbb{R})$ and a *generating set* for $\mathcal{P}_2(\mathbb{R})$.

Linear independence. Setting

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = \mathbf{0},$$

where the zero vector $\mathbf{0}$ of $\mathcal{P}_2(\mathbb{R})$ is given by the zero polynomial (i.e., $\mathbf{0}(x) = 0 + 0x + 0x^2$, for $x \in \mathbb{R}$), we obtain that

$$\alpha_1(1) + \alpha_2(x-2) + \alpha_3(x-2)^2 = \mathbf{0}(x)$$

for any $x \in \mathbb{R}$, which upon rearrangement yields

$$(\alpha_1 - 2\alpha_2 + 4\alpha_3) + (\alpha_2 - 4\alpha_3)x + \alpha_3 x^2 = \mathbf{0}(x) = 0 + 0x + 0x^2.$$

Since the left-hand side polynomial equals the zero polynomial on the right-hand side, then their respective coefficients must match; hence we have that

$$\begin{cases} \alpha_1 - 2\alpha_2 + 4\alpha_3 = 0 \\ \alpha_2 - 4\alpha_3 = 0 \\ \alpha_3 = 0 \end{cases}$$

Solving the above system directly yields that we must have that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Thus the set $\{f_1, f_2, f_3\}$ is linearly independent.

Note: An alternative solution is to note that we must have

$$\alpha_1(1) + \alpha_2(x-2) + \alpha_3(x-2)^2 = 0$$

for any $x \in \mathbb{R}$. So we can plug different values of x and solve for the scalar coefficients:

$$\begin{cases} x = 2: \quad \alpha_1 = 0 \\ x = 0: \quad \alpha_1 - 2\alpha_2 + 4\alpha_3 = 0 \\ x = 1: \quad \alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases}$$

Solving the above system similarly yields that we must have that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

and hence the set $\{f_1, f_2, f_3\}$ is linearly independent.

<u>Generating set.</u> We need to show that any polynomial function f in $\mathcal{P}_2(\mathbb{R})$ can be written as a linear combination of f_1, f_2 and f_3. In other words, for any $f(x) = a_0 + a_1 x + a_2 x^2$ in $\mathcal{P}_2(\mathbb{R})$ (with given real coefficients a_0, a_1 and a_2), we need to find scalars α, β and γ (in terms of a_0, a_1 and a_2) such that

$$f(x) = \alpha f_1(x) + \beta f_2(x) + \gamma f_3(x),$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 = \alpha(1) + \beta(x-2) + \gamma(x-2)^2.$$

After grouping terms in the right-hand side above, we obtain that

$$a_0 + a_1 x + a_2 x^2 = (\alpha - 2\beta + 4\gamma) + (\beta - 4\gamma)x + \gamma x^2.$$

The left- and right-hand side polynomials being equal means that they must match in their respective coefficients. Thus:

$$\begin{cases} a_0 = \alpha - 2\beta + 4\gamma \\ a_1 = \beta - 4\gamma \\ a_2 = \gamma \end{cases}$$

We thus have a linear system with three equations in the three unknowns α, β and γ. Solving this system yields that

$$\begin{cases} \gamma = a_2 \\ \beta = a_1 + 4\gamma = a_1 + 4a_2 \\ \alpha = a_0 + 2\beta - 4\gamma = a_0 + 2(a_1 + 4a_2) - 4(a_2) = a_0 + 2a_1 + 4a_2 \end{cases} \tag{4}$$

Thus $f(x) = a_0 + a_1x + a_2x^2$ in $\mathcal{P}_2(\mathbb{R})$ can indeed be written as a linear combination of $f_1(x)$, $f_2(x)$ and $f_3(x)$:

$$f(x) = (a_0 + 2a_1 + 4a_2)f_1(x) + (a_1 + 4a_2)f_2(x) + a_2f_2(x)$$

and $\{f_1, f_2, f_3\}$ is a generating set for $\mathcal{P}_2(\mathbb{R})$. Hence $\mathcal{B} = (f_1, f_2, f_3)$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

b) Since basis $\mathcal{B} = (f_1, f_2, f_3)$ for $\mathcal{P}_2(\mathbb{R})$ has three vectors, then $\mathcal{P}_2(\mathbb{R})$ has dimension 3.

c) The polynomial $f(x) = 9 + x + 3x^2$ has coefficients $a_0 = 9$, $a_1 = 1$ and $a_2 = 3$. Then directly using these values in (4) yields the coordinates of $f(x) = 9 + x + 3x^2$ in the basis $\mathcal{B} = \{f_1, f_2, f_3\}$:

$$\begin{cases} \gamma = a_2 = 3 \\ \beta = a_1 + 4a_2 = 1 + 4(3) = 13 \\ \alpha = a_0 + 2a_1 + 4a_2 = 9 + 2(1) + 4(3) = 23 \end{cases}$$

Thus the *coordinate vector* of $f(x) = 9 + x + 3x^2$ under the basis $\mathcal{B} = (f_1, f_2, f_3)$ is

$$(\alpha, \beta, \gamma)_{\mathcal{B}} = (23, 13, 3)_{\mathcal{B}}.$$