Queen's University - APSC174 **Winter 2025**

Tutorial 08

1. Show which of the following mappings between real vector spaces is linear and which is not linear:
   (a) $L : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $L((x, y)) = (x + y,\ x - y,\ xy)$.
   (b) $L : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $L((x, y)) = (x + y,\ x - y,\ x)$.

2. Consider the linear transformation $L\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by

$$L(x, y, z, w) = (x - y + 4z + 4w,\, 3x + y + 4z + 8w,\, x + 4y - 6z - w).$$

   (a) Which of $\mathbf{v}_1 = (3, -1, 0, -1)$, $\mathbf{v}_2 = (1, 4, 3, -2)$, and $\mathbf{v}_3 = (2, -2, -1, 0)$ (if any) are in $\mathrm{Ker}(L)$?
   (b) Is $\mathbf{v} = 3\mathbf{v}_1 - 4\mathbf{v}_3 = (1, 5, 4, -3)$ in $\mathrm{Ker}(L)$?
   (c) Find two linearly independent vectors in $\mathrm{Ker}(L)$, and so show that $\dim(\mathrm{Ker}(L))$ is at least 2.

3. Consider the linear transformation $L\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$L(x, y, z) = (x - y + 7z,\, 3x + y + 9z,\, x + 4y - 8z).$$

   (a) Which of $\mathbf{w}_1 = (1, 7, 6)$, $\mathbf{w}_2 = (2, 2, -3)$, and $\mathbf{w}_3 = (1, 4, 5)$ (if any) are in $\mathrm{Im}(L)$?

   NOTES: (1) Deciding if a vector is in $\mathrm{Im}(L)$ can be expressed as the problem of "does this system of linear equations have a solution?", a problem we know how to answer by the RREF method of solving linear equations. (2) Rather than doing the RREF procedure three times, perhaps there is a way to answer all three questions at the same time by putting all three coefficient vectors on the right side of the line in the augmented matrix, and going through the RREF procedure only once.

   (b) Is $\mathbf{w}_4 = \frac{1}{3}\mathbf{w}_1 + \frac{1}{3}\mathbf{w}_2 = (1, 3, 1)$ in $\mathrm{Im}(L)$?
   (c) Is $\mathbf{w}_5 = 2\mathbf{w}_1 - 3\mathbf{w}_3$ in $\mathrm{Im}(L)$?
   (d) Find two linearly independent vectors in $\mathrm{Im}(L)$, and so show that $\dim(\mathrm{Im}(L))$ is at least 2.

4. Let $L\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be given by the rule

$$L(f) = 9f + f'',$$

where f'' means the second derivative of f.
   (a) Show that L is a linear transformation.
   (b) Show that both $\sin(3x)$ and $\cos(3x)$ are in $\mathrm{Ker}(L)$.
   (c) Show that $\alpha_1 \sin(3x) + \alpha_2 \cos(3x)$ is in $\mathrm{Ker}(L)$ for any real α_1 and α_2.
   (d) Are $\sin(3x)$ and $\cos(3x)$ linearly independent? (If so, this shows that the dimension of $\mathrm{Ker}(L)$ is at least 2.)

5. Suppose that $L\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and we know that $L(5, 3) = (2, 5, 4)$ and $L(3, 2) = (1, 2, 3)$.
   (a) Write $(1, 0)$ as a linear combination of $(5, 3)$ and $(3, 2)$. Also write $(0, 1)$ as a linear combination of $(5, 3)$ and $(3, 2)$.
   (b) Using your answers from (a), and the information about L, determine $L(1, 0)$ and $L(0, 1)$.
   (c) Use your answers from (b) to deduce $L(7, 5)$.
   (d) Use your answers from (b) to find a formula for $L(x, y)$.
   (e) Find the standard matrix for L.