1. Show which of the following mappings between real vector spaces is linear and which is not linear:
   (a) $L : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $L((x, y)) = (x + y,\ x - y,\ xy)$.
   (b) $L : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $L((x, y)) = (x + y,\ x - y,\ x)$.

Solution.

(a) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L((x, y)) = (x + y,\ x - y,\ xy)$, $\forall (x, y) \in \mathbb{R}^2$. The question is: Is L a **linear** mapping ? Let $\alpha = 2$, and let $\mathbf{v} = (1, 1)$. We have:

$$\begin{aligned} L(\alpha\mathbf{v}) &= L(2(1,1)) = L((2,2)) = (4,0,4), \\ \alpha L(\mathbf{v}) &= 2L((1,1)) = 2(2,0,1) = (4,0,2), \end{aligned}$$

which shows that for this particular choice of α and $\mathbf{v}$, we have $L(\alpha\mathbf{v}) \neq \alpha L(\mathbf{v})$. Hence, L is **not** linear.

(b) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L((x, y)) = (x + y,\ x - y,\ x)$, $\forall (x, y) \in \mathbb{R}^2$. The question is: Is L a **linear** mapping ? We have, $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $\forall \alpha \in \mathbb{R}$:

$$\begin{aligned} L((x_1,y_1)+(x_2,y_2)) &= L((x_1+x_2, y_1+y_2)) \\ &= ((x_1+x_2)+(y_1+y_2), (x_1+x_2)-(y_1+y_2), x_1+x_2) \\ &= ((x_1+y_1)+(x_2+y_2), (x_1-y_1)+(x_2-y_2), x_1+x_2) \\ &= (x_1+y_1, x_1-y_1, x_1)+(x_2+y_2, x_2-y_2, x_2) \\ &= L((x_1,y_1))+L((x_2,y_2)), \\ L(\alpha(x_1,y_1)) &= L((\alpha x_1, \alpha y_1)) \\ &= (\alpha x_1+\alpha y_1, \alpha x_1-\alpha y_1, \alpha x_1) \\ &= \alpha(x_1+y_1, x_1-y_1, x_1) \\ &= \alpha L((x_1,y_1)), \end{aligned}$$

and this proves that L is **linear**.

2. Consider the linear transformation $L\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by

$$L(x, y, z, w) = (x - y + 4z + 4w,\ 3x + y + 4z + 8w,\ x + 4y - 6z - w).$$

   (a) Which of $\mathbf{v}_1 = (3, -1, 0, -1)$, $\mathbf{v}_2 = (1, 4, 3, -2)$, and $\mathbf{v}_3 = (2, -2, -1, 0)$ (if any) are in $\mathrm{Ker}(L)$?
   (b) Is $\mathbf{v} = 3\mathbf{v}_1 - 4\mathbf{v}_3 = (1, 5, 4, -3)$ in $\mathrm{Ker}(L)$?
   (c) Find two linearly independent vectors in $\mathrm{Ker}(L)$, and so show that $\dim(\mathrm{Ker}(L))$ is at least 2.

Solution.

(a) Recall that a vector $\mathbf{v}$ is in $\mathrm{Ker}(L)$ if and only $L(\mathbf{v}) = \mathbf{0}$.

Since

$$\begin{aligned} L(\mathbf{v}_1) &= \Big(3-(-1)+4\cdot 0+4\cdot(-1),\ 3\cdot 3+(-1)+4\cdot 0+8\cdot(-1), 3+4\cdot(-1)-6\cdot 0-(-1)\Big) \\ &= \Big(3+1-4,\ 9-1-8,\ 3-4+1\Big) = (0,0,0) = \mathbf{0} \in \mathbb{R}^3, \end{aligned}$$

the vector $\mathbf{v}_1$ is in $\mathrm{Ker}(L)$.

Since

$$\begin{aligned} L(\mathbf{v}_2) &= \Big(1-4+4\cdot 3+4\cdot(-2),\ 3\cdot 1+3+4\cdot 3+8\cdot(-2), 1+4\cdot 4-6\cdot 3-(-2)\Big) \\ &= \Big(1-4+12-8 3+4+12-16,\ 1+16-18+2\Big) = (1,3,1) \neq \mathbf{0}, \end{aligned}$$

the vector $\mathbf{v}_2$ is *not* in $\mathrm{Ker}(L)$.

Finally, since

$$\begin{aligned} L(\mathbf{v}_3) &= \Big(2-(-2)+4\cdot(-1)+4\cdot 0,\ 3\cdot 2+(-2)+4\cdot(-1)+8\cdot 0, 2+4\cdot(-2)-6\cdot(-1)-0\Big) \\ &= \Big(2+2-4,\ 6-2-4,\ 2-8-6\Big) = (0,0,0) = \mathbf{0} \in \mathbb{R}^3, \end{aligned}$$

the vector $\mathbf{v}_3$ is in $\mathrm{Ker}(L)$.

(b) Here are two possible solutions.

Solution 1. We know from class that for any linear transformation $L\colon \mathbf{V} \longrightarrow \mathbf{W}$ the set $\mathrm{Ker}(L)$ is a subspace of $\mathbf{V}$. In our case, this means that $\mathrm{Ker}(L)$ is a subspace of $\mathbb{R}^4$.

From part (a) we know that both $\mathbf{v}_1$ and $\mathbf{v}_3$ are in $\mathrm{Ker}(L)$, and since $\mathrm{Ker}(L)$ is a subspace, every linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$ is also in $\mathrm{Ker}(L)$. In particular, the linear combination $3\mathbf{v}_1 - 4\mathbf{v}_3$ must also be in $\mathrm{Ker}(L)$.

Solution 2. We apply the test to be in $\mathrm{Ker}(L)$. Since

$$\begin{aligned} L(\mathbf{v}) &= \Big(1-5+4\cdot 4+4\cdot(-3),\ 3\cdot 1+5+4\cdot 4+8\cdot(-3), 1+4\cdot 5-6\cdot 4-(-3)\Big) \\ &= \Big(1-5+16-12,\ 3+5+16-24,\ 1+20-24+3\Big) = (0,0,0) = \mathbf{0} \in \mathbb{R}^3, \end{aligned}$$

the vector $\mathbf{v}$ is in $\mathrm{Ker}(L)$.

(c) We have already found two vectors in $\mathrm{Ker}(L)$, namely $\mathbf{v}_1$ and $\mathbf{v}_3$. These two vectors are linearly independent. To see this we consider the equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_3 = \mathbf{0}$, that is :

$$\alpha_1(3,-1,0,-1) + \alpha_2(2,-2,-1,0) = (3\alpha_1+2\alpha_2,\ -\alpha_1-2\alpha_2,\ -\alpha_2, -\alpha_1) = \mathbf{0} = (0,0,0,0).$$

Looking at the last two coordinates gives the equations $-\alpha_2 = 0$ and $-\alpha_1 = 0$, whose only solution is $\alpha_1 = 0$ and $\alpha_2 = 0$. Therefore $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly independent.

Why does finding two linearly independent elements in $\mathrm{Ker}(L)$ show that $\dim(\mathrm{Ker}(L))$ is at least 2?

Here is the argument : Let $d = \dim(\mathrm{Ker}(L))$. By definition, this means that $\mathrm{Ker}(L)$ has a basis with d elements, say $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_d\}$. The set $\{\mathbf{v}_1, \mathbf{v}_3\}$ is in $\mathrm{Ker}(L)$ and linearly independent, and $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ is (among other things) a spanning set for $\mathrm{Ker}(L)$. By the Key Lemma, the number of elements in the linearly independent set $\{\mathbf{v}_1, \mathbf{v}_3\}$ is less than or equal to number of elements in the generating set $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$. I.e., $2 \leqslant d$. Since $d = \dim(\mathrm{Ker}(L))$, this is saying that $\dim(\mathrm{Ker}(L)) \geqslant 2$.

More generally, this kind of argument shows that if we find k linearly independent vectors in some vector space $\mathbf{W}$, then $k \leqslant \dim(\mathbf{W})$.

3. Consider the linear transformation $L\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$L(x,y,z) = (x-y+7z,\ 3x+y+9z,\ x+4y-8z).$$

(a) Which of $\mathbf{w}_1 = (1, 7, 6)$, $\mathbf{w}_2 = (2, 2, -3)$, and $\mathbf{w}_3 = (1, 4, 5)$ (if any) are in $\text{Im}(L)$?

NOTES: (1) Deciding if a vector is in $\text{Im}(L)$ can be expressed as the problem of "does this system of linear equations have a solution?", a problem we know how to answer by the RREF method of solving systems of linear equations. (2) Rather than doing the RREF procedure three times, perhaps there is a way to answer all three questions at the same time by putting all three coefficient vectors on the right side of the line in the augmented matrix, and going through the RREF procedure only once.

(b) Is $\mathbf{w}_4 = \frac{1}{3}\mathbf{w}_1 + \frac{1}{3}\mathbf{w}_2 = (1, 3, 1)$ in $\text{Im}(L)$?

(c) Is $\mathbf{w}_5 = 2\mathbf{w}_1 - 3\mathbf{w}_3$ in $\text{Im}(L)$?

(d) Find two linearly independent vectors in $\text{Im}(L)$, and so show that $\dim(\text{Im}(L))$ is at least 2.

Solution.

(a) Asking if $\mathbf{w}_1 = (1, 7, 6)$ is in $\text{Im}(L)$ is asking if there are $(x, y, z) \in \mathbb{R}^3$ so that

$$L(x, y, z) = (x - y + 7z,\ 3x + y + 9z,\ x + 4y - 8z) = (1, 7, 6),$$

that is, asking if the system of linear equations

$$\begin{array}{rcrcrcr} x & - & y & + & 7z & = & 1 \\ 3x & + & y & + & 9z & = & 7 \\ x & + & 4y & - & 8z & = & 6 \end{array}$$

has a solution. This is a problem we know how to solve using the RREF method. Writing down the augmented matrix which corresponds to the system and row reducing we get

$$\left[\begin{array}{rrr|r} 1 & -1 & 7 & 1 \\ 3 & 1 & 9 & 7 \\ 1 & 4 & -8 & 6 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is no row of the form $\left[\begin{array}{rrr|r} 0 & 0 & 0 & 1 \end{array}\right]$, this system of linear equations has a solution. The RREF algorithm shows us that the general solution is $(x, y, z) = (2, 1, 0) + t \cdot (-4, 3, 1)$. In particular, setting $t = 0$, $(x, y, z) = (2, 1, 0)$ is a solution, as we can check by putting this back into the formula for L :

$$\begin{aligned} L(2, 1, 0) &= \Big(2 - 1 + 7 \cdot 0,\ 3 \cdot 2 + 1 + 9 \cdot 0,\ 2 + 4 \cdot 1 - 8 \cdot 0\Big) \\ &= (1, 7, 6). \end{aligned}$$

Since there is a vector we can put into L so that the output is $(1, 7, 6)$ (equivalently, the system of linear equations above has a solution), $(1, 7, 6) \in \text{Im}(L)$.

Similarly to decide if $\mathbf{w}_2 = (2, 2, -3)$ is in $\text{Im}(L)$, we express the problem as whether the system of linear equations

$$\begin{array}{rcrcrcr} x & - & y & + & 7z & = & 2 \\ 3x & + & y & + & 9z & = & 2 \\ x & + & 4y & - & 8z & = & -3 \end{array}$$

has a solution, which we solve by using the RREF method :

$$\left[\begin{array}{rrr|r} 1 & -1 & 7 & 2 \\ 3 & 1 & 9 & 2 \\ 1 & 4 & -8 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 4 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From the RREF, we see that a solution exists (for example, $(x, y, z) = (1, -1, 0)$ is a solution), so that $(2, 2, -3) \in \text{Im}(L)$.

Finally, we apply the same method to $\mathbf{w}_3 = (1, 4, 5)$, converting the problem of whether $(1, 4, 5) \in \text{Im}(L)$ to the problem of whether the system

$$\begin{array}{rcrcrcr} x & - & y & + & 7z & = & 1 \\ 3x & + & y & + & 9z & = & 4 \\ x & + & 4y & - & 8z & = & 5 \end{array}$$

has a solution. Using the RREF method, we get

$$\left[\begin{array}{rrr|r} 1 & -1 & 7 & 1 \\ 3 & 1 & 9 & 4 \\ 1 & 4 & -8 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Since there is a row of the form $\left[\begin{array}{rrr|r} 0 & 0 & 0 & 1 \end{array}\right]$, this system of linear equations has no solution. Therefore there is no $(x, y, z) \in \mathbb{R}^3$ so that $L(x, y, z) = (1, 4, 5)$, and $(1, 4, 5)$ is not in $\text{Im}(L)$.

Rather than do these three RREF calculations separately, we could have done them all at once, starting with the matrix

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 7 & 1 & 2 & 1 \\ 3 & 1 & 9 & 7 & 2 & 4 \\ 1 & 4 & -8 & 6 & -3 & 5 \end{array}\right]$$

which has, on the left of the $|$ line, the coefficients of x, y, and z in the linear equations, and on the right of the line all the different possible solutions we are trying to test. Row reducing this matrix, we get

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 7 & 1 & 2 & 1 \\ 3 & 1 & 9 & 7 & 2 & 4 \\ 1 & 4 & -8 & 6 & -3 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|rrr} 1 & 0 & 4 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The way to interpret this matrix is that it gives us the three different answers to the three different row reduction problems all at once. To understand those answers, it is easier to separate them back into the three different problems (by using the part of the matrix to the left of the $|$ line and, one at a time, each of the columns to the right of the line). This gives the three matrices

$$\left[\begin{array}{rrr|r} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{rrr|r} 1 & 0 & 4 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{and} \quad \left[\begin{array}{rrr|r} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

we saw above. Looking at the matrices separately we see that the first two vectors are in $\text{Im}(L)$, while the third is not.

(b) Here are two solutions to this question.

Solution 1. We know from class that for any linear transformation $L\colon \mathbf{V} \longrightarrow \mathbf{W}$ the set $\text{Im}(L)$ is a subspace of $\mathbf{W}$. In our case, this means that $\text{Im}(L)$ is a subspace of $\mathbb{R}^3$.

From part (a) we know that $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $\text{Im}(L)$. As in 1(b), since $\text{Im}(L)$ is a subspace, every linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$ is in $\text{Im}(L)$. In particular, $\mathbf{w}_4 = \frac{1}{2}\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_2 \in \text{Im}(L)$.

Solution 2. We apply the method of part (a) to each of the two questions. For $\mathbf{w}_4 = (1, 3, 1)$ the RREF problem is

$$\left[\begin{array}{rrr|r} 1 & -1 & 7 & 1 \\ 3 & 1 & 9 & 3 \\ 1 & 4 & -8 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 4 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The RREF form shows that there is a solution. In particular, it gives the solution $(x, y, z) = (1, 0, 0)$ which was obvious anyway : the first column *is* $(1, 3, 1)$.

(c) We can also use both methods used in (b) to solve (c).

Solution 1. We again use the fact that $\text{Im}(L)$ is a subspace of $\mathbb{R}^3$.

From part (a) we know that $\mathbf{w}_1$ is in $\text{Im}(L)$ and that $\mathbf{w}_3$ is not. Using this and an argument with linear combinations shows that $\mathbf{w}_5 = 2\mathbf{w}_1 - 3\mathbf{w}_3$ *cannot* be in $\text{Im}(L)$. Specifically, if $\mathbf{w}_5$ were in $\text{Im}(L)$, then the linear combination

$$\frac{1}{3}\Big(2\mathbf{w}_1 - \mathbf{w}_5\Big) = \frac{1}{3}\Big(2\mathbf{w}_1 - (2\mathbf{w}_1 - 3\mathbf{w}_3)\Big) = \frac{1}{3}\Big(3\mathbf{w}_3\Big) = \mathbf{w}_3$$

would also be in $\text{Im}(L)$. Since we know that $\mathbf{w}_3 \notin \text{Im}(L)$, this means that we must also have $\mathbf{w}_5 \notin \text{Im}(L)$.

Solution 2. For $\mathbf{w}_5 = 2\mathbf{w}_1 - 3\mathbf{w}_3 = (-1, 2, -3)$ the RREF problem is

$$\left[\begin{array}{rrr|r} 1 & -1 & 7 & -1 \\ 3 & 1 & 9 & 2 \\ 1 & 4 & -8 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

which shows that the system of equation has no solution, and so $\mathbf{w}_5 \notin \mathrm{Im}(L)$.

(d) In part (a) we have seen that the two vectors $\mathbf{w}_1 = (1, 7, 6)$ and $\mathbf{w}_2 = (2, 2, -3)$ are in $\mathrm{Im}(L)$. We now check that they are linearly independent. The equation $\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 = \mathbf{0}$ is

$$\alpha_1(1, 7, 6) + \alpha_2(2, 2, -3) = (\alpha_1 + 2\alpha_2,\ 7\alpha_1 + 2\alpha_2,\ 6\alpha_1 - 3\alpha_2) = (0, 0, 0) = \mathbf{0},$$

which is the same as the system of linear equations

$$\begin{array}{rcrcl} \alpha_1 & + & 2\alpha_2 & = & 0 \\ 7\alpha_1 & + & 2\alpha_2 & = & 0 \\ 6\alpha_1 & - & 3\alpha_2 & = & 0 \end{array}$$

Writing the system as an augmented matrix and row reducing, we get

$$\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 7 & 2 & 0 \\ 6 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is $(\alpha_1, \alpha_2) = (0, 0)$ with no free parameters, i.e., $\alpha_1 = 0$, $\alpha_2 = 0$ is the unique solution, and so $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly independent. Finally, using a similar argument as in question 2.(c), we conclude that $\dim(\mathrm{Im}(L)) \geq 2$.

4. Let $L \colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be given by the rule

$$L(f) = 9f + f'',$$

where f'' means the second derivative of f.

(a) Show that L is a linear transformation.

(b) Show that both $\sin(3x)$ and $\cos(3x)$ are in $\mathrm{Ker}(L)$.

(c) Show that $\alpha_1 \sin(3x) + \alpha_2 \cos(3x)$ is in $\mathrm{Ker}(L)$ for any real α_1 and α_2.

(d) Are $\sin(3x)$ and $\cos(3x)$ linearly independent? (If so, this shows that the dimension of $\mathrm{Ker}(L)$ is at least 2.)

Solution.

(a) We check the two tests to be a linear transformation.

Addition test: For all $f_1, f_2 \in C^\infty(\mathbb{R})$, is $L(f_1 + f_2) = L(f_1) + L(f_2)$?

Using the fact that $(f_1 + f_2)'' = f_1'' + f_2''$, that is, that the second derivative of a sum is the sum of the second derivatives, we calculate that

$$L(f_1 + f_2) = 9(f_1 + f_2) + (f_1 + f_2)'' = 9f_1 + 9f_2 + f_1'' + f_2'' = (9f_1 + f_1'') + (9f_2 + f_2'') = L(f_1) + L(f_2),$$

so L passes the addition test.

Scalar multiplication test: For all $f \in C^\infty(\mathbb{R})$, $\alpha \in \mathbb{R}$, is $L(\alpha \cdot f) = \alpha \cdot L(f)$?

Using the fact that for a second-differentiable function f, and real number α we have $(\alpha \cdot f)'' = \alpha \cdot f''$, we calculate that

$$L(\alpha \cdot f) = 9(\alpha \cdot f) - (\alpha \cdot f)'' = \alpha(9f) - \alpha f'' = \alpha(9f - f'') = \alpha \cdot L(f),$$

so L passes the scalar multiplication test.

Since L passes both tests, L is a linear transformation.

(b) The second derivatives of $\sin(3x)$ and $\cos(3x)$ are :

$$\begin{array}{ccccc} \sin(3x) & \xmapsto{\frac{d}{dx}} & 3\cos(3x) & \xmapsto{\frac{d}{dx}} & -9\sin(3x) \\ \cos(3x) & \xmapsto{\frac{d}{dx}} & -3\sin(3x) & \xmapsto{\frac{d}{dx}} & -9\cos(3x). \end{array}$$

Therefore,

$$L\Big(\sin(3x)\Big) = 9\Big(\sin(3x)\Big) + \Big(\sin(3x)\Big)'' = 9\sin(3x) + \Big(-9\sin(3x)\Big) = \mathbf{0},$$

and

$$L\Big(\cos(3x)\Big) = 9\Big(\cos(3x)\Big) + \Big(\cos(3x)\Big)'' = 9\cos(3x) + \Big(-9\cos(3x)\Big) = \mathbf{0}.$$

In the two equations above $\mathbf{0}$ denotes the zero vector in $C^\infty(\mathbb{R})$, i.e., the constant zero function $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$. This shows that both $\sin(3x)$ and $\cos(3x)$ are in $\mathrm{Ker}(L)$.

(c) Here are two solutions to this problem.

Solution 1. We know from class that for any linear transformation $L\colon \mathbf{V} \longrightarrow \mathbf{W}$ the set $\mathrm{Ker}(L)$ is a subspace of $\mathbf{V}$. In our case, this means that $\mathrm{Ker}(L)$ is a subspace of $C^\infty(\mathbb{R})$.

From part (a) we know that both $\sin(3x)$ and $\cos(3x)$ are in $\mathrm{Ker}(L)$, and since $\mathrm{Ker}(L)$ is a subspace every linear combination of $\sin(3x)$ and $\cos(3x)$ is also in $\mathrm{Ker}(L)$. Therefore, for any α_1, $\alpha_2 \in \mathbb{R}$, $\alpha_1\sin(3x) + \alpha_2\cos(3x)$ is in $\mathrm{Ker}(L)$.

Solution 2. For any α_1, $\alpha_2 \in \mathbb{R}$ we have

$$\begin{aligned} \Big(\alpha_1\sin(3x) + \alpha_2\cos(3x)\Big)'' &= \alpha_1\sin(3x)'' + \alpha_2\cos(3x)'' \\ &= \alpha_1\Big(-9\sin(3x)\Big) + \alpha_2\Big(9\cos(3x)\Big) \\ &= -9\alpha_1\sin(3x) - 9\alpha_2\sin(3x). \end{aligned}$$

Therefore

$$\begin{aligned} L\Big(\alpha_1\sin(3x) + \alpha_2\cos(3x)\Big) &= 9\Big(\alpha_1\sin(3x) + \alpha_2\cos(3x)\Big) + \Big(\alpha_1\sin(3x) + \alpha_2\cos(3x)\Big)'' \\ &= 9\alpha_1\sin(3x) + 9\alpha_2\cos(3x) - 9\alpha_1\sin(3x) - 9\alpha_2\cos(3x) \\ &= \mathbf{0}. \end{aligned}$$

So $\alpha_1\sin(3x) + \alpha_2\cos(3x)$ is in $\mathrm{Ker}(L)$.

(d) We now show that $\sin(3x)$ and $\cos(3x)$ are linearly independent. Suppose that we have α_1 and α_2 so that

$$\alpha_1\sin(3x) + \alpha_2\cos(3x) = \mathbf{0}$$

Since this is an equality of functions, we are saying that the equality holds for all $x \in \mathbb{R}$. Plugging in $x = 0$ we get

$$0 = \alpha_1\sin(0) + \alpha_2\cos(0) = \alpha_1 \cdot 0 + \alpha_2 \cdot 1 = \alpha_2.$$

On the other hand, plugging in $x = \frac{\pi}{6}$, we get

$$0 = \alpha_1\sin(\pi/2) + \alpha_2\cos(\pi/2) + \alpha_1 \cdot 1 + \alpha_2 \cdot 0 = \alpha_1.$$

Starting with the equation $\alpha_1\sin(3x) + \alpha_2\cos(3x) = \mathbf{0}$, we see that this implies that $\alpha_1 = 0$ and $\alpha_2 = 0$. So, $\alpha_1 = 0$ and $\alpha_2 = 0$ is the only solution, and so $\Big\{\sin(3x), \cos(3x)\Big\}$ is linearly independent. Hence by a similar argument as in question 2.(c), we conclude that $\dim(\mathrm{Ker}(L)) \geq 2$.

5. Suppose that $L\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and we know that $L(5,3) = (2,5,4)$ and $L(3,2) = (1,2,3)$.

(a) Write $(1,0)$ as a linear combination of $(5,3)$ and $(3,2)$. Also write $(0,1)$ as a linear combination of $(5,3)$ and $(3,2)$.

(b) Using your answers from (a), and the information about L, determine $L(1,0)$ and $L(0,1)$.

(c) Use your answers from (b) to deduce $L(7,5)$.

(d) Use your answers from (b) to find a formula for $L(x,y)$.

(e) Find the standard matrix for L.

Solution.

(a) To write $(1,0)$ as a linear combination of $(5,3)$ and $(3,2)$ we need to solve

$$(1,0) = \alpha_1(5,3) + \alpha_2(3,2) = (5\alpha_1 + 3\alpha_2,\ 3\alpha_1 + 2\alpha_2)\,,$$

which is the same as the system of linear equations

$$\begin{array}{rcrcl} 5\alpha_1 & + & 3\alpha_2 & = & 1 \\ 3\alpha_1 & + & 2\alpha_2 & = & 0. \end{array}$$

We can solve this system by writing down the corresponding augmented matrix and row-reducing :

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 3 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

Therefore the unique solution is $\alpha_1 = 2$, $\alpha_2 = -3$, i.e., $(1,0) = 2(5,3) - 3(3,2)$. Note that the columns of the original matrix are the vectors involved in the linear combination problem, something we have seen from class.

Similarly, to write $(0,1)$ as a linear combination of $(5,3)$ and $(3,2)$ we write down the corresponding matrix and row reduce :

$$\left[\begin{array}{cc|c} 5 & 3 & 0 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array}\right]$$

Therefore $(0,1) = -3(5,3) + 5(3,2)$.

NOTE: As in question 3(a), we could have solved both equations at the same time by combining the augmented matrices and row reducing, as shown below.

$$\left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 5 \end{array}\right].$$

As before, to understand what the answer means it might be easiest to think of it as giving the two different results of row reducing, namely

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array}\right].$$

(b) In this question we use the *Key Property of linear transformations*, if $L\colon \mathbf{V} \longrightarrow \mathbf{W}$ is a linear transformation, and if we know $L(\mathbf{v}_1)$, $L(\mathbf{v}_2)$,..., $L(\mathbf{v}_p)$ where $\mathbf{v}_1,\ldots,\mathbf{v}_p \in \mathbf{V}$, then we can deduce $L(\mathbf{v})$ for all $\mathbf{v}$ in the span of $\mathbf{v}_1,\ldots,\mathbf{v}_p$. For, if $\mathbf{v}$ is in the span of $\mathbf{v}_1,\ldots,\mathbf{v}_p$, we can write $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p$ and then

$$(*) \qquad L(\mathbf{v}) = L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_p L(\mathbf{v}_p).$$

The second equality uses the fact that L is a linear transformation, i.e., that L passes the addition and scalar multiplication tests.

In our case, since we know from part (a) that $(1,0) = 2(5,3) - 3(3,2)$, and that $L(5,3) = (2,5,4)$ and $L(3,2) = (1,2,3)$, we can deduce that

$$\begin{aligned} L(1,0) &= L\Big(2(5,3) - 3(3,2)\Big) = 2L(5,3) - 3L(3,2) \\ &= 2(2,5,4) - 3(1,2,3) = (1,4,-1). \end{aligned}$$

Similarly, since $(0,1) = -3(5,3) + 5(3,2)$, we deduce that

$$\begin{aligned} L(0,1) &= L\Big(-3(5,3) + 5(3,2)\Big) = -3L(5,3) + 5L(3,2) \\ &= -3(2,5,4) + 5(1,2,3) = (-1,-5,3). \end{aligned}$$

(c) Since $(7,5) = 7(1,0) + 5(0,1)$, using the answer from (b) and the key property of linear transformations, we have

$$\begin{aligned} L(7,5) &= L\Big(7(1,0) + 5(0,1)\Big) = 7L(1,0) + 5L(0,1) \\ &= 7(1,4,-1) + 5(-1,-5,3) = (2,3,8). \end{aligned}$$

(d) Since $(x,y) = x(1,0) + y(0,1)$, using the answer from (b) and the key property of linear transformations, we have

$$\begin{aligned} L(x,y) &= L\Big(x(1,0) + y(0,1)\Big) = xL(1,0) + yL(0,1) \\ &= x(1,4,-1) + y(-1,-5,3) = (x-y,\ 4x-5y,\ -x+3y). \end{aligned}$$

(e) By definition, the standard matrix for a linear transformation $L\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the 3×2 matrix whose first column is $L(1,0)$ and whose second column is $L(0,1)$. In (b) we have computed that $L(1,0) = (1,4,-1)$ and $L(0,1) = (-1,-5,3)$, and so the standard matrix for L is

$$\begin{bmatrix} 1 & -1 \\ 4 & -5 \\ -1 & 3 \end{bmatrix}.$$

NOTE: Once we have the standard matrix, it is easy to answer questions (c) and (d).

$$L(7,5) = \begin{bmatrix} 1 & -1 \\ 4 & -5 \\ -1 & 3 \end{bmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + 5\begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix},$$

and

$$L(x,y) = \begin{bmatrix} 1 & -1 \\ 4 & -5 \\ -1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} x-y \\ 4x-5y \\ -x+3y \end{pmatrix}.$$