1. Let $L\colon \mathbf{V} \longrightarrow \mathbf{W}$ be a linear transformation, with $\mathbf{V}$ and $\mathbf{W}$ finite dimensional vector spaces.

   (a) Prove that L is surjective if and only if $\dim(\mathrm{Im}(L)) = \dim(\mathbf{W})$. NOTES: (1) Since the question is an "if and only if", there are two directions to prove; make sure you understand what each of the directions is. In one of the directions you will need the result of **Tutorial 7**, **5(b)**.

   (b) Show that L is injective if and only if $\dim(\mathrm{Ker}(L)) = 0$.

Solution.

(a) The two directions are :

(a1) If L is surjective, prove that $\dim(\mathrm{Im}(L)) = \dim(\mathbf{W})$; and

(a2) If $\dim(\mathrm{Im}(L)) = \dim(\mathbf{W})$, prove that L is surjective.

The arguments for proving these are :

Proof of (a1): By definition, a map $f\colon X \longrightarrow Y$ is surjective when everything in Y is in the image of f, i.e., when $\mathrm{Im}(f) = Y$. Applied to a linear transformation $L\colon \mathbf{V} \longrightarrow \mathbf{W}$, this means that L is surjective when $\mathrm{Im}(L) = \mathbf{W}$. But, if these vector spaces are equal, then their dimensions are also equal.

Proof of (a2): We know from class that $\mathrm{Im}(L)$ is a subspace of $\mathbf{W}$. We are also assuming that $\dim(\mathrm{Im}(L)) = \dim(\mathbf{W})$. By **Homework 7**, **5(b)** if one finite dimensional vector space is contained in another and their dimensions are equal, then the vector spaces are also equal. Therefore, $\mathrm{Im}(L) = \mathbf{W}$, and so L is surjective.

(b) This part also has two directions :

(b1) If L is injective, show that $\dim(\mathrm{Ker}(L)) = 0$; and

(b2) If $\dim(\mathrm{Ker}(L)) = 0$ show that L is injective.

In class we have learned that L is injective if and only if $\mathrm{Ker}(L) = \{\mathbf{0}\}$. The dimension of the vector space $\{\mathbf{0}\}$ is zero.

Conversely, the only zero-dimensional vector space is $\{\mathbf{0}\}$. To see this suppose that $\mathbf{V}$ is a zero-dimensional vector space. Like every vector space, $\mathbf{V}$ contains a zero vector. If $\mathbf{V}$ contained any other vector, then that non-zero vector by itself would be a linearly independent set, and so $\dim(\mathbf{V}) \geqslant 1$. Thus, if $\dim(\mathbf{V}) = 0$, there can be no other vector in $\mathbf{V}$, so $\mathbf{V} = \{\mathbf{0}\}$.

Putting these two facts together proves the statement. Explicitly,

Proof of (b1): If L is injective then $\mathrm{Ker}(L) = \{\mathbf{0}\}$, so $\dim(\mathrm{Ker}(L)) = \dim(\{\mathbf{0}\}) = 0$.

Proof of (b2): If $\dim(\mathrm{Ker}(L)) = 0$ then $\mathrm{Ker}(L) = \{\mathbf{0}\}$, so L is injective.

A linear transformation $L\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given in each of Problems 2–5 below. For each of them :

(a) Find the standard matrix for L.

(b) Find a basis for $\mathrm{Im}(L)$.

(c) Find a basis for $\mathrm{Ker}(L)$.

(d) Find $\dim(\mathrm{Im}(L))$.

(e) Find $\dim(\mathrm{Ker}(L))$.

(f) What does the Rank-Nullity theorem predict for L? Does it hold in this case?

(g) Is L injective?

(h) Is L surjective?

For parts (g) and (h) you should use the results of Problem 1.

2. $L_1 \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ is the linear transformation given by

$$L_1(x, y, z) = (2x + y - z,\ -x + 2y - 7z,\ 3y - 9z,\ -4x + y - 7z).$$

Solution.

(a) Since $L_1(1, 0, 0) = (2, -1, 0, -4)$, $L_1(0, 1, 0) = (1, 2, 3, 1)$, and $L_1(0, 0, 1) = (-1, -7, -9, -7)$, the standard matrix for L_1 is

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & -7 \\ 0 & 3 & -9 \\ -4 & 1 & -7 \end{bmatrix}$$

(b) The first step in finding a basis for the kernel or image of L_1 is to put the standard matrix into RREF. The RREF of A is :

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & -7 \\ 0 & 3 & -9 \\ -4 & 1 & -7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A possible sequence of row operations giving this RREF is included at the end of the homework assigment.

Since the leading ones appear in the first and second columns of the RREF, by our algorithm from class the first and second columns of A, the standard matrix for L_1, form a basis for $\mathrm{Im}(L_1)$. Thus

$$((2, -1, 0, -4),\ (1, 2, 3, 1))$$

is a basis for $\mathrm{Im}(L_1)$.

(c) Adding an extra column of zeros to the RREF we get the matrix

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Using the algorithm to parameterize all solutions to the corresponding system of equations we get the general solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix},$$

with one free parameter. By our algorithm for finding a basis for $\mathrm{Ker}(L_1)$, the list $((-1, 3, 1))$, consisting only of the vector $(-1, 3, 1)$, is a basis for $\mathrm{Ker}(L_1)$.

(d) $\dim(\mathrm{Im}(L_1)) = 2$, since the basis for $\mathrm{Im}(L_1)$ in (b) has 2 vectors.

(e) $\dim(\mathrm{Ker}(L_1)) = 1$, since the basis for $\mathrm{Ker}(L_1)$ in (c) has 1 vector.

(f) The rank-nullity theorem predicts that $\dim(\mathrm{Ker}(L_1)) + \dim(\mathrm{Im}(L_1)) = \dim(\mathbb{R}^3) = 3$. This holds in this Problem, since (by (d) and (e)),

$$\dim(\mathrm{Ker}(L_1)) + \dim(\mathrm{Im}(L_1)) = 1 + 2 = 3.$$

(g) No, since $\dim(\mathrm{Ker}(L_1)) = 1 \neq 0$, we see by 1(b) that L_1 is not injective.

(h) No, since $\dim(\mathrm{Im}(L_1)) = 2 \neq 4 = \dim(\mathbb{R}^4)$, we see by 1(a) that L_1 is not surjective.

3. $L_2 \colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is the linear transformation given by

$$L_2(x, y, z, w) = (2x - y + 4z + w,\ x + 3y + 9z + 2w,\ 2y + 4z + 2w).$$

Solution.

(a) Since $L_2(1, 0, 0, 0) = (2, 1, 0)$, $L_2(0, 1, 0, 0) = (-1, 3, 2)$, $L_2(0, 0, 1, 0) = (4, 9, 4)$, and $L_2(0, 0, 0, 1) = (1, 2, 2)$, the standard matrix for L_2 is

$$B = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 3 & 9 & 2 \\ 0 & 2 & 4 & 2 \end{bmatrix}.$$

(b) The RREF of B is

$$\begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 3 & 9 & 2 \\ 0 & 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A possible sequence of row operations giving this RREF is included at the end of the homework assigment.

The leading ones are in the first, second, and fourth column of the RREF. Therefore the first, second, and fourth columns of B are a basis for $\mathrm{Im}(L_2)$. In this case a basis for $\mathrm{Im}(L_2)$ is

$$((2, 1, 0),\ (-1, 3, 2),\ (1, 2, 2)).$$

(c) Adding a zero column to the RREF we get the matrix

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The algorithm for finding the general solution to the associated system of linear equations gives the general solution as

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = t \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix},$$

with one free parameter. By our algorithm $((-3, -3, 1, 0))$ is therefore a basis for $\mathrm{Ker}(L_2)$.

(d) $\dim(\mathrm{Im}(L_2)) = 3$, since the basis for $\mathrm{Im}(L_2)$ in (b) has 3 vectors.

(e) $\dim(\mathrm{Ker}(L_2)) = 1$, since the basis for $\mathrm{Ker}(L_2)$ in (c) has 1 vector.

(f) The rank-nullity theorem predicts that $\dim(\mathrm{Ker}(L_2)) + \dim(\mathrm{Im}(L_2)) = \dim(\mathbb{R}^4) = 4$. This holds in this Problem, since (by (d) and (e)),

$$\dim(\mathrm{Ker}(L_2)) + \dim(\mathrm{Im}(L_2)) = 1 + 3 = 4.$$

(g) No, since $\dim(\mathrm{Ker}(L_2)) = 1 \neq 0$, we see by 1(b) that L_2 is not injective.

(h) Yes, since $\dim(\mathrm{Im}(L_2)) = 3 = \dim(\mathbb{R}^3)$, we see by 1(a) that L_2 is surjective.

4. $L_3 \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the linear transformation given by

$$L_3(x, y, z) = (3x + 2y + 2z,\ -x + 3y + 3z,\ y + z).$$

Solution.

(a) Since $L_3(1,0,0) = (3,-1,0)$, $L_3(0,1,0) = (2,3,1)$, and $L(0,0,1) = (2,3,1)$, the standard matrix for L_3 is

$$C = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) The RREF of C is

$$\begin{bmatrix} 3 & 2 & 2 \\ -1 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A possible sequence of row operations giving this RREF is included at the end of the homework assigment.

The first and second columns of C have leading ones, so the first and second columns of C form a basis for $\mathrm{Im}(L_3)$, i.e.,

$$((3,-1,0),\ (2,3,1))$$

is a basis for $\mathrm{Im}(L_3)$.

(c) Adding a zero column to the RREF we get the matrix

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

The general solution to the corresponding system of equations is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

with one free parameter. By our algorithm $((0,-1,1))$ is therefore a basis for $\mathrm{Ker}(L_3)$.

(d) $\dim(\mathrm{Im}(L_3)) = 2$, since the basis for $\mathrm{Im}(L_3)$ in (b) has 2 vectors.

(e) $\dim(\mathrm{Ker}(L_3)) = 1$, since the basis for $\mathrm{Ker}(L_3)$ in (c) has 1 vector.

(f) The rank-nullity theorem predicts that $\dim(\mathrm{Ker}(L_3)) + \dim(\mathrm{Im}(L_3)) = \dim(\mathbb{R}^3) = 3$. This holds in this Problem, since (by (d) and (e)),

$$\dim(\mathrm{Ker}(L_3)) + \dim(\mathrm{Im}(L_3)) = 1 + 2 = 3.$$

(g) No, since $\dim(\mathrm{Ker}(L_3)) = 1 \neq 0$, we see by 1(b) that L_3 is not injective.

(h) No, since $\dim(\mathrm{Im}(L_3)) = 2 \neq \dim(\mathbb{R}^3)$, we see by 1(a) that L_3 is not surjective.

5. $L_4 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the linear transformation given by

$$L_4(x, y) = (2x + 3y,\, x + y).$$

Solution.

(a) Since $L_4(1, 0) = (2, 1)$ and $L_4(0, 1) = (3, 1)$ the standard matrix for L_4 is

$$D = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

(b) The RREF of D is

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A possible sequence of row operations giving this RREF is included at the end of the homework assigment.

Since all rows of the RREF have leading ones, the rows of D form a basis for $\mathrm{Im}(L_4)$, i.e.,

$$((2, 1),\, (3, 1))$$

is a basis for $\mathrm{Im}(L_4)$.

(c) Adding a zero column to the RREF we get the matrix

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

The unique solution to the corresponding system of equations is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are no free parameters in this system, and so the algorithm for finding a basis for $\mathrm{Ker}(L_4)$ gives the empty set (i.e., $\emptyset$) as the basis.

(d) $\dim(\mathrm{Im}(L_4)) = 2$, since the basis for $\mathrm{Im}(L_4)$ in (b) has 2 vectors.

(e) $\dim(\mathrm{Ker}(L_4)) = 0$, since the basis for $\mathrm{Ker}(L_4)$ in (c) has 0 vectors.

(f) The rank-nullity theorem predicts that $\dim(\mathrm{Ker}(L_4)) + \dim(\mathrm{Im}(L_4)) = \dim(\mathbb{R}^2) = 2$. This holds in this Problem, since (by (d) and (e)),

$$\dim(\mathrm{Ker}(L_4)) + \dim(\mathrm{Im}(L_4)) = 0 + 2 = 2.$$

(g) Yes, since $\dim(\mathrm{Ker}(L_4)) = 0$, we see by 1(b) that L_4 is injective.

(h) Yes, since $\dim(\mathrm{Im}(L_4)) = 2 = \dim(\mathbb{R}^2)$, we see by 1(a) that L_4 is surjective.